

On the Correlation Functions of the Characteristic Polynomials of Non-Hermitian Random Matrices with Independent Entries

Ie. Afanasiev*

Abstract

The paper is concerned with the asymptotic behavior of the correlation functions of the characteristic polynomials of non-Hermitian random matrices with independent entries. It is shown that the correlation functions behave like that for the Complex Ginibre Ensemble up to a factor depending only on the fourth absolute moment of the common probability law of the matrix entries.

1 Introduction

The Random Matrix Theory has been developed for some sixty years. The story began with the study of symmetric and Hermitian random matrices. They have remained the most studied ever since. However, non-Hermitian matrices are not so well studied.

The present paper is concerned with the simplest non-Hermitian ensemble which is an analog of the Wigner ensemble. The matrices are constructed of independent identically distributed (i.i.d.) complex random variables. More precisely, the matrices have the form

$$M_n = \frac{1}{\sqrt{n}}X = \frac{1}{\sqrt{n}}(x_{jk})_{j,k=1}^n, \quad (1.1)$$

where x_{jk} are i.i.d. complex random variables such that

$$\mathbf{E}\{x_{jk}\} = \mathbf{E}\{x_{jk}^2\} = 0, \quad \mathbf{E}\{|x_{jk}|^2\} = 1. \quad (1.2)$$

Here and everywhere below \mathbf{E} denotes the expectation with respect to (w.r.t.) all random variables. This ensemble has various applications in physics, neuroscience, economics, etc. For detailed information see [1] and references therein.

Define the Normalized Counting Measure (NCM) of eigenvalues as

$$N_n(\Delta) = \#\{\lambda_j^{(n)} \in \Delta, j = 1, \dots, n\}/n,$$

where Δ is an arbitrary Borel set in the complex plane, $\{\lambda_j^{(n)}\}_{j=1}^n$ are the eigenvalues of M_n . The NCM is known to converge to the uniform distribution on the unit disc. This distribution is called the circular law. This result has a long and rich history. Mehta was the first who obtained it for x_{jk} being complex Gaussian in 1967 [26]. The proof strongly relied on

*B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, Kharkiv, Ukraine; e-mail: afanasiev@ilt.kharkov.ua. Supported in part by The President of Ukraine grant and by the Akhiezer Foundation scholarship.

the explicit formula for the common probability density of eigenvalues due to Ginibre [17]. Unfortunately, there is no such a formula in the general case. That is why other methods have to be used. The Hermitization approach introduced by Girko [18] appeared to be an effective method. The main idea is to reduce the study of matrices (1.1) to the study of Hermitian matrices using the logarithmic potential of a measure

$$P_\mu(z) = \int_{\mathbb{C}} \log |z - \zeta| d\mu(\zeta).$$

This approach was successfully developed by Girko in the next series of works [19, 20, 21, 22]. The final result in the most general case was established by Tao and Vu [38]. Notice that there are a lot of partial results besides those listed above. The interested reader is directed to [4].

The Central Limit Theorem (CLT) for non-Hermitian random matrices linear statistics was proven in some partial cases in [12, 30, 31]. The best results for today were obtained by Kopel in [25] for smooth functions and by Tao and Vu in [39] for small radii disc indicators. Both mentioned results require $\Re x_{jk}$ and $\Im x_{jk}$ being independent and having the first four moments as in the Gaussian case (which is often referred as GinUE similarly to the Gaussian Unitary Ensemble (GUE) in Hermitian case). The article [39] also deals with a local regime for these matrices. It was established that under the same conditions the k -point correlation function converges in vague topology to that for GinUE.

One can observe that non-Hermitian random matrices are more complicated than their Hermitian counterparts. Indeed, the Hermitian case was successfully dealt with using the Stieltjes transform or the moments method. However, a measure in the plane can not be recovered from its Stieltjes transform or its moments. Thus these approaches to the analysis fail in the non-Hermitian case.

The present article suggests to apply the supersymmetry technique (SUSY). It is a rather powerful method which is widely applied at the physical level of rigor (for instance [15, 28]). There are also a lot of rigorous results, which were obtained using SUSY in the recent years, e.g. [8], [9], [32], [33], [34], etc. Supersymmetry technique is usually used in order to obtain an integral representation for ratios of determinants. Since the main spectral characteristics such as density of states, spectral correlation functions, etc. often can be expressed via ratios of determinants, SUSY allows to get the integral representation for these characteristics too. For detailed discussion on connection between spectral characteristics and ratios of determinants see [37, 5, 23]. See also [16, 29].

Let us consider the second spectral correlation function R_2 defined by the equality

$$\mathbf{E} \left\{ 2 \sum_{1 \leq j_1 < j_2 \leq n} \eta(\lambda_{j_1}^{(n)}, \lambda_{j_2}^{(n)}) \right\} = \int_{\mathbb{C}^2} \eta(\lambda_1, \lambda_2) R_2(\lambda_1, \lambda_2) d\bar{\lambda}_1 d\lambda_1 d\bar{\lambda}_2 d\lambda_2,$$

where the function $\eta: \mathbb{C}^2 \rightarrow \mathbb{C}$ is bounded, continuous and symmetric in its arguments. Using the logarithmic potential, R_2 can be represented via ratios of the determinants of M_n with the most singular term of the form

$$\int_0^{\varepsilon_0} \int_0^{\varepsilon_0} \frac{\partial^2}{\partial \delta_1 \partial \delta_2} \mathbf{E} \left\{ \prod_{j=1}^2 \frac{\det((M_n - z_j)(M_n - z_j)^* + \delta_j)}{\det((M_n - z_j)(M_n - z_j)^* + \varepsilon_j)} \right\} \Big|_{\delta=\varepsilon} d\varepsilon_1 d\varepsilon_2 \quad (1.3)$$

The integral representation for (1.3) obtained by SUSY will contain both commuting and anti-commuting variables. Such type integrals are rather difficult to analyse. That is why

one would investigate a more simple but similar integral to shed light on the situation. This integral arises from the study of the correlation functions of the characteristic polynomials. Moreover, the correlation functions of the characteristic polynomials are of independent interest. They were studied for many ensembles of Hermitian and real symmetric matrices, for instance [6], [7], [35], [36], [33] etc. The other result on the asymptotic behavior of the correlation functions of the characteristic polynomials of non-Hermitian matrices of the form $H + i\Gamma$, where H is from GUE and Γ is a fixed matrix of rank M , was obtained in [14]. The kernel computed there, in the limit of rank $M \rightarrow \infty$ of the perturbation Γ (taken after matrix size $n \rightarrow \infty$) after appropriate rescaling approaches the form (1.9). It was demonstrated in [13, Sec. 2.2].

Let us introduce the m^{th} correlation function of the characteristic polynomials

$$f_m(Z) = \mathbf{E} \left\{ \prod_{j=1}^m \det(M_n - z_j) (M_n - z_j)^* \right\}, \quad (1.4)$$

where

$$Z = \text{diag}\{z_1, \dots, z_m\} \quad (1.5)$$

and z_1, \dots, z_m are complex parameters which may depend on n . We are interested in the asymptotic behavior of (1.4), as $n \rightarrow \infty$, for

$$z_j = z_0 + \frac{\zeta_j}{\sqrt{n}}, \quad j = 1, 2, \dots, m, \quad (1.6)$$

where $z_0, \zeta_1, \dots, \zeta_m$ are n -independent complex numbers, and z_0 is in the bulk of the spectrum, i.e. $|z_0| < 1$. For GinUE the value of (1.4) is known. In [2] Akemann and Vernizzi showed that

$$\mathbf{E} \left\{ \prod_{j=1}^m \det(X - z_j^{(1)}) (X - z_j^{(2)})^* \right\} = \left(\prod_{l=n}^{n+m-1} l! \right) \frac{\det(K_n(z_j^{(1)}, \bar{z}_k^{(2)}))_{j,k=1}^m}{\Delta(Z^{(1)})\Delta((Z^{(2)})^*)}, \quad (1.7)$$

where

$$K_n(z, w) = \sum_{l=0}^{n+m-1} \frac{(z\bar{w})^l}{l!}.$$

and $\Delta(Z^{(1)})$ (resp. $\Delta((Z^{(2)})^*)$) is a Vandermonde determinant of $z_1^{(1)}, \dots, z_m^{(1)}$ (resp. $\bar{z}_1^{(2)}, \dots, \bar{z}_m^{(2)}$). Putting $z_j^{(1)} = z_j^{(2)} = \sqrt{n}z_j$, z_j of the form (1.6), one deduces from (1.7) that

$$\lim_{n \rightarrow \infty} n^{-\frac{m^2-m}{2}} \frac{f_m(Z)}{f_1(z_1) \cdots f_1(z_m)} = \frac{\det(K(\zeta_j, \zeta_k))_{j,k=1}^m}{|\Delta(\mathcal{Z})|^2}, \quad (1.8)$$

where $\mathcal{Z} = \text{diag}\{\zeta_1, \dots, \zeta_m\}$ and

$$K(z, w) = e^{-|z|^2/2 - |w|^2/2 + z\bar{w}}. \quad (1.9)$$

The other result on the characteristic polynomials of GinUE matrices was obtained by Webb and Wong in [41]. They showed that for any complex γ with $\Re\gamma > -2$

$$\mathbf{E} \{ |\det(M_n - z_0)|^\gamma \} = n^{\frac{\gamma^2}{8}} e^{\frac{\gamma}{2}n(|z_0|^2-1)} \frac{(2\pi)^{\frac{\gamma}{4}}}{G(1 + \frac{\gamma}{2})} (1 + o(1)), \quad (1.10)$$

where G is the Barnes G -function.

In this article the general case of arbitrary distribution, satisfying (1.2), is considered. The main result of the paper is

Theorem 1. Let an ensemble of non-Hermitian random matrices M_n be defined by (1.1) and (1.2). Let also the first $2m$ absolute moments of the common distribution of entries of M_n be finite and z_j , $j = 1, \dots, m$, have the form (1.6). Then

(i) the m^{th} correlation function of the characteristic polynomials (1.4) satisfies the asymptotic relation

$$\lim_{n \rightarrow \infty} n^{-\frac{m^2-m}{2}} \frac{f_m(Z)}{f_1(z_1) \cdots f_1(z_m)} = C_{m,z_0}^{(i)} e^{\frac{m^2-m}{2} (1-|z_0|^2)^2 \kappa_{2,2}} \frac{\det(K(\zeta_j, \zeta_k))_{j,k=1}^m}{|\Delta(Z)|^2},$$

where $C_{m,z_0}^{(i)}$ is some constant, which does not depend on the common distribution of entries and on ζ_1, \dots, ζ_m ; $\kappa_{2,2} = \mathbf{E}\{|x_{11}|^4\} - 2$ and $K(z, w)$ is defined in (1.9);

(ii) in particular case $\zeta_1 = \dots = \zeta_m = 0$ we have

$$\mathbf{E} \left\{ |\det(M_n - z_0)|^{2m} \right\} = C_{m,z_0}^{(ii)} e^{\frac{m^2-m}{2} (1-|z_0|^2)^2 \kappa_{2,2}} n^{\frac{m^2}{2}} e^{mn(|z_0|^2-1)} (1 + o(1)), \quad (1.11)$$

where $C_{m,z_0}^{(ii)}$ is some constant, which does not depend on the common distribution of entries.

Remark 1. Going through the proof of Theorem 1 one can determine constants $C_{m,z_0}^{(i)}$ and $C_{m,z_0}^{(ii)}$. Their values are

$$C_{m,z_0}^{(i)} = 1, \quad C_{m,z_0}^{(ii)} = (2\pi)^{m/2} \left(\prod_{j=1}^{m-1} j! \right)^{-1}.$$

Let us point out that $\kappa_{2,2} = 0$ in Gaussian case, and the result of Theorem 1 is in full agreement with the results for GinUE (cf. (1.8), (1.10)). Theorem also shows that the asymptotics of f_m^1 is similar to the asymptotics of the m -point spectral correlation function (see [39]).

The paper is organized as follows. Section 2 is devoted to the derivation of the suitable integral representation for f_m by using the SUSY approach. In Section 3 we apply the steepest descent method to the obtained integral representation and find out the asymptotic behavior of f_m . In order to compute it, the Harish-Chandra/Itsykson–Zuber formula is used. For the reader convenience both sections are divided into two parts treating the Gaussian and general cases respectively.

Acknowledgement. The author is grateful to Prof. M. Shcherbina for the statement of the problem and fruitful discussions.

1.1 Notations

Through out the article lower-case letters denote scalars, bold lower-case letters denote vectors, upper-case letters denote matrices and bold upper-case letters denote sets of matrices. We use the same letter for a matrix, for its columns and for its entries. Table 1 shows an exact correspondence. Besides, for any matrix A we denote by $(A)_j$ its j -th column and by $(A)_{kj}$ its entry in the k -th row and in the j -th column.

The term ‘‘Grassmann variable’’ is a synonym for ‘‘anti-commuting variable’’. The variables of integration $\phi, \varphi, \theta, \vartheta, \rho, \xi, \tau$ and ν are Grassmann variables, all the other variables of

¹Here and below we omit Z only if $Z = \text{diag}\{z_1, \dots, z_m\}$.

Set of matrices	Matrix	Column	Entry
\mathbf{Q}	$Q_{p,s}$		$q_{\alpha\beta}^{(p,s)}$
Ξ	$\Xi_{p,s}$		$\xi_{\alpha\beta}^{(p,s)}$
	Φ	ϕ_j	ϕ_{kj}
	Θ	θ_j	θ_{kj}
	$Y_{k,p,s}$		$y_{\alpha\beta}^{(k,p,s)}$
	U		u_{kj}
	V		v_{kj}

Table 1: Notation correspondence

integration unspecified by an integration domain are either complex or real. We split all the generators of Grassmann algebra into two equal sets and consider the generators from the second set as “conjugates” of that from the first set. I.e., for Grassmann variable v we use v^* to denote its “conjugate”. Furthermore, if $\Upsilon = (v_{jk})$ means a matrix of Grassmann variables then Υ^+ is a matrix (v_{kj}^*) . d -dimensional vectors are identified with $d \times 1$ matrices.

Integrals without limits denote either integration over Grassmann variables or integration over the whole space \mathbb{C}^d or \mathbb{R}^d . Let also $d\mathbf{t}^*d\mathbf{t}$ ($\mathbf{t} = (t_1, \dots, t_d)^T \in \mathbb{C}^d$) denote the measure $\prod_{j=1}^d d\bar{t}_j dt_j$ on the space \mathbb{C}^d . Similarly, for vectors with anti-commuting entries $d\boldsymbol{\tau}^+d\boldsymbol{\tau} = \prod_{j=1}^d d\tau_j^* d\tau_j$. Note that the space of matrices is a linear space over \mathbb{C} . Thus the same notations are used for them as well.

Through out the article $U(m)$ is a group of unitary $m \times m$ matrices. In order to simplify the notation we sometimes write Q_j instead of $Q_{j,j}$ and $q_{\alpha\beta}^{(j)}$ instead of $q_{\alpha\beta}^{(j,j)}$. In addition, C, C_1 denote various n -independent constants which can be different in different formulas.

2 Integral representation for \mathbf{f}_m

In this section we obtain a convenient integral representation for the correlation function of characteristic polynomials \mathbf{f}_m defined by (1.4).

Proposition 1. *Let an ensemble M_n be defined by (1.1) and (1.2). Then the m^{th} correlation function of the characteristic polynomials \mathbf{f}_m defined by (1.4) can be represented in the following form*

$$\mathbf{f}_m = \left(\frac{n}{\pi}\right)^{c_m} \int g(\mathbf{Q}) e^{(n-c_m)f(\mathbf{Q})} d\mathbf{Q}, \quad (2.1)$$

where $c_m = 2^{2m-1}$, $\mathbf{Q} = (Q_{p,s})_{p,s=1}^m$ with even $p+s$, $Q_{p,s}$ is a complex $\binom{m}{p} \times \binom{m}{s}$ matrix,

$$d\mathbf{Q} = \prod_{\substack{p+s \text{ is even} \\ 0 \leq p, s \leq m}} dQ_{p,s}^* dQ_{p,s} \text{ and}$$

$$f(\mathbf{Q}) = - \sum_{\substack{p+s \text{ is even} \\ 0 \leq p, s \leq m}} \text{tr } Q_{p,s}^* Q_{p,s} + \log h(\mathbf{Q}); \quad (2.2)$$

$$g(\mathbf{Q}) = (h(\mathbf{Q})^{c_m} + n^{-1/2} \mathbf{p}_a(\mathbf{Q})) \exp \left\{ -c_m \sum_{\substack{p+s \text{ is even} \\ 0 \leq p, s \leq m}} \text{tr } Q_{p,s}^* Q_{p,s} \right\};$$

$$h(\mathbf{Q}) = \det A + n^{-1/2} \tilde{h}(Q_2) + n^{-1} \mathbf{p}_c(\hat{\mathbf{Q}}); \quad (2.3)$$

$$A = A(Q_1) = \begin{pmatrix} -Z & Q_1 \\ -Q_1^* & -Z^* \end{pmatrix} \quad (2.4)$$

with $\mathbf{p}_a(\mathbf{Q})$, $\mathbf{p}_c(\hat{\mathbf{Q}})$ and $\tilde{h}(Q_2)$ being certain polynomials specified in the proof below, and $\hat{\mathbf{Q}}$ containing all $Q_{p,s}$ except Q_1 .

Remark 2. Let $Q_1 = U\Lambda V^*$ be the singular value decomposition of the matrix Q_1 , i.e. $\Lambda = \text{diag}\{\lambda_j\}_{j=1}^m$, $\lambda_j \geq 0$, $U, V \in U(m)$. In order to perform asymptotic analysis let us change the variables $Q_1 = U\Lambda V^*$ in (2.1). Since the Jacobian is $\frac{2^m \pi^{m^2}}{(\prod_{j=1}^{m-1} j!)^2} \Delta^2(\Lambda^2) \prod_{j=1}^m \lambda_j$ (see e.g. [24]) we obtain

$$\begin{aligned} \mathbf{f}_m &= C n^{c_m} \int_{\mathcal{D}} \Delta^2(\Lambda^2) \prod_{j=1}^m \lambda_j \left[g_0(\Lambda, \hat{\mathbf{Q}}) + \frac{1}{\sqrt{n}} g_r(U\Lambda V^*, \hat{\mathbf{Q}}) \right] \\ &\quad \times \exp \left\{ (n - c_m) \left[f_0(\Lambda, \hat{\mathbf{Q}}) + \frac{1}{\sqrt{n}} f_r(U\Lambda V^*, \hat{\mathbf{Q}}) \right] \right\} d\mu(U) d\mu(V) d\Lambda d\hat{\mathbf{Q}}, \end{aligned} \quad (2.5)$$

where $\mathcal{D} = \{(\Lambda, U, V, \hat{\mathbf{Q}}) \mid \lambda_j \geq 0, j = 1, \dots, m, U, V \in U(m)\}$, μ is a Haar measure, $d\Lambda = \prod_{j=1}^m d\lambda_j$ and

$$f_0(\mathbf{Q}) = - \sum_{\substack{p+s \text{ is even} \\ 0 \leq p, s \leq m}} \text{tr } Q_{p,s}^* Q_{p,s} + \log h_0(Q_1); \quad (2.6)$$

$$g_0(\mathbf{Q}) = h_0(Q_1)^{c_m} \exp \left\{ -c_m \sum_{\substack{p+s \text{ is even} \\ 0 \leq p, s \leq m}} \text{tr } Q_{p,s}^* Q_{p,s} \right\} = e^{c_m f_0(\mathbf{Q})};$$

$$h_0(Q_1) = \det \left(A + \frac{1}{\sqrt{n}} \begin{pmatrix} Z & 0 \\ 0 & Z^* \end{pmatrix} \right) = \prod_{j=1}^m (|z_0|^2 + \lambda_j^2); \quad (2.7)$$

$$\begin{aligned} f_r(\mathbf{Q}) &= \sqrt{n}(f(\mathbf{Q}) - f_0(\mathbf{Q})); \\ g_r(\mathbf{Q}) &= \sqrt{n}(g(\mathbf{Q}) - g_0(\mathbf{Q})). \end{aligned} \quad (2.8)$$

Notice that $f_0(U\Lambda V^*, \hat{\mathbf{Q}}) = f_0(\Lambda, \hat{\mathbf{Q}})$ and the same for g_0 .

Remark 3. In the special case $m = 1$ we have

$$\mathbf{f}_1(z) = \frac{n}{\pi} \int \exp \left\{ n(-|q|^2 + \log(|z|^2 + |q|^2)) \right\} d\bar{q} dq.$$

Changing variables to polar coordinates and performing a simple Laplace integration, we obtain

$$f_1(z) = 2n \int_0^{+\infty} r \exp \left\{ n(-r^2 + \log(|z|^2 + r^2)) \right\} dr = \sqrt{2\pi n} e^{n(|z|^2 - 1)} (1 + o(1)). \quad (2.9)$$

Remark 4. In the Gaussian case the representations (2.1) and (2.5) become much more simple and have the form

$$\begin{aligned} f_m &= \left(\frac{n}{\pi}\right)^{m^2} \int e^{nf(Q_1)} dQ_1^* dQ_1 \\ &= Cn^{m^2} \int_{\mathbb{R}_+^m} \int_{U(m)} \int_{U(m)} \Delta^2(\Lambda^2) \prod_{j=1}^m \lambda_j \times e^{nf(U\Lambda V^*)} d\mu(U) d\mu(V) d\Lambda, \end{aligned} \quad (2.10)$$

where

$$f(Q_1) = -\text{tr } Q_1^* Q_1 + \log \det A. \quad (2.11)$$

2.1 Proof of Proposition 1

The proof is strongly relied on the SUSY techniques. A reader who is not familiar with Grassmann variables can find all the necessary facts in [10] or [11]. For more serious introduction to SUSY see [3].

The key formulas of the subsection are well-known Gaussian integration formula

$$\int_{\mathbb{C}^n} \exp \{ -\mathbf{t}^* B \mathbf{t} - \mathbf{t}^* \mathbf{h}_2 - \mathbf{h}_1^* \mathbf{t} \} d\mathbf{t}^* d\mathbf{t} = \pi^n \det^{-1} B \exp \{ \mathbf{h}_1^* B^{-1} \mathbf{h}_2 \}, \quad (2.12)$$

valid for any positive definite matrix B and even Grassmann variables (i.e. sums of products of even number of Grassmann variables) $\mathbf{h}_1, \mathbf{h}_2$, and its Grassmann analog

$$\int \exp \{ -\boldsymbol{\tau}^+ B \boldsymbol{\tau} - \boldsymbol{\tau}^+ \mathbf{v}_2 - \mathbf{v}_1^+ \boldsymbol{\tau} \} d\boldsymbol{\tau}^+ d\boldsymbol{\tau} = \det B \exp \{ \mathbf{v}_1^+ B^{-1} \mathbf{v}_2 \} \quad (2.13)$$

valid for arbitrary complex matrix B and odd Grassmann variables (i.e. sums of products of odd number of Grassmann variables) $\mathbf{v}_1^+, \mathbf{v}_2$. Rewrite the expression (1.4) for f_m using (2.13) and (1.1)

$$f_m = \mathbf{E} \left\{ \int \exp \left\{ -\sum_{j=1}^m \phi_j^+ \left(\frac{1}{\sqrt{n}} X - z_j \right) \phi_j - \sum_{j=1}^m \boldsymbol{\theta}_j^+ \left(\frac{1}{\sqrt{n}} X - z_j \right)^* \boldsymbol{\theta}_j \right\} d\Phi d\Theta \right\},$$

where $\phi_j, \boldsymbol{\theta}_j, j = 1, \dots, m$ are n -dimensional vectors with components ϕ_{kj} and θ_{kj} respectively, $d\Phi = \prod_{j=1}^m d\phi_j^+ d\phi_j$ and $d\Theta = \prod_{j=1}^m d\boldsymbol{\theta}_j^+ d\boldsymbol{\theta}_j$. The terms in the exponent can be rearranged

as following

$$\begin{aligned}
-\sum_{j=1}^m \phi_j^+ X \phi_j &= -\text{tr } \Phi^+ X \Phi = \text{tr } \Phi \Phi^+ X = \sum_{k,l=1}^n (\Phi \Phi^+)_{lk} x_{kl}, \\
-\sum_{j=1}^m \theta_j^+ X^* \theta_j &= -\text{tr } \Theta^+ X^* \Theta = \text{tr } \Theta \Theta^+ X^* = \sum_{k,l=1}^n (\Theta \Theta^+)_{kl} \bar{x}_{kl}, \\
\sum_{j=1}^m \phi_j^+ z_j \phi_j &= \sum_{j=1}^m \sum_{k=1}^n \phi_{kj}^* z_j \phi_{kj} = \sum_{k=1}^n \sum_{j=1}^m \phi_{kj}^* z_j \phi_{kj} = \sum_{k=1}^n \varphi_k^+ Z \varphi_k, \\
\sum_{j=1}^m \theta_j^+ \bar{z}_j \theta_j &= \sum_{j=1}^m \sum_{k=1}^n \theta_{kj}^* \bar{z}_j \theta_{kj} = \sum_{k=1}^n \sum_{j=1}^m \theta_{kj}^* \bar{z}_j \theta_{kj} = \sum_{k=1}^n \vartheta_k^+ Z^* \vartheta_k,
\end{aligned}$$

where Θ and Φ are matrices composed of columns $\theta_1, \dots, \theta_m$ and ϕ_1, \dots, ϕ_m respectively, $\varphi_k = (\Phi^T)_k$, $\vartheta_k = (\Theta^T)_k$, Z is defined in (1.5). Hence

$$\begin{aligned}
f_m = \mathbf{E} \left\{ \int \exp \left\{ \sum_{k=1}^n \varphi_k^+ Z \varphi_k + \sum_{k=1}^n \vartheta_k^+ Z^* \vartheta_k \right. \right. \\
\left. \left. + \frac{1}{\sqrt{n}} \sum_{k,l=1}^n (\Phi \Phi^+)_{lk} x_{kl} + \frac{1}{\sqrt{n}} \sum_{k,l=1}^n (\Theta \Theta^+)_{kl} \bar{x}_{kl} \right\} d\Phi d\Theta \right\}. \quad (2.14)
\end{aligned}$$

To simplify the reading, the remaining steps are first explained in the case when the entries of X are Gaussian.

2.1.1 Gaussian case

Taking the expectation in (2.14) we get

$$f_m = \int \exp \left\{ \sum_{k=1}^n \varphi_k^+ Z \varphi_k + \sum_{k=1}^n \vartheta_k^+ Z^* \vartheta_k + \sum_{k,l=1}^n \frac{1}{n} (\Phi \Phi^+)_{lk} (\Theta \Theta^+)_{kl} \right\} d\Phi d\Theta.$$

Notice that

$$\sum_{k,l=1}^n (\Phi \Phi^+)_{lk} (\Theta \Theta^+)_{kl} = \text{tr } \Phi \Phi^+ \Theta \Theta^+ = -\text{tr } \Theta^T (\Phi^T)^+ \Phi^T (\Theta^T)^+.$$

Then the Hubbard–Stratonovich transformation is applied. The transformation is an application of (2.12) in the reverse direction. It yields

$$\begin{aligned}
f_m = \left(\frac{n}{\pi} \right)^{m^2} \int \exp \left\{ \sum_{k=1}^n \varphi_k^+ Z \varphi_k + \sum_{k=1}^n \vartheta_k^+ Z^* \vartheta_k + \text{tr } \Theta^T (\Phi^T)^+ Q_1 \right. \\
\left. - \text{tr } Q_1^* \Phi^T (\Theta^T)^+ - n \text{tr } Q_1^* Q_1 \right\} d\Phi d\Theta dQ_1^* dQ_1, \quad (2.15)
\end{aligned}$$

where Q_1 is a $m \times m$ matrix. Transforming the terms

$$\begin{aligned}
\text{tr } \Theta^T (\Phi^T)^+ Q_1 &= -\text{tr} (\Phi^T)^+ Q_1 \Theta^T = -\sum_{k=1}^n \varphi_k^+ Q_1 \vartheta_k, \\
\text{tr } Q_1^* \Phi^T (\Theta^T)^+ &= -\text{tr} (\Theta^T)^+ Q_1^* \Phi^T = -\sum_{k=1}^n \vartheta_k^+ Q_1^* \varphi_k,
\end{aligned}$$

one can rewrite (2.15) in the form

$$f_m = \left(\frac{n}{\pi}\right)^{m^2} \int dQ_1^* dQ_1 e^{-n \operatorname{tr} Q_1^* Q_1} \prod_{k=1}^n \int e^{-\rho_k^+ A \rho_k} d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k,$$

where A is defined in (2.4) and

$$\rho_k = \begin{pmatrix} \varphi_k \\ \vartheta_k \end{pmatrix}. \quad (2.16)$$

Finally, integration via (2.13) leads us to (2.10).

2.1.2 General case

In order to treat the general case let us introduce a notation for a kind of ‘‘Laplace–Fourier transform’’

$$\psi(t_1, t_2) := \mathbf{E} \left\{ e^{t_1 x_{11} + t_2 \bar{x}_{11}} \right\}.$$

Then the expectation in (2.14) can be written in the following form

$$\begin{aligned} f_m &= \int \prod_{k,l=1}^n \psi \left(\frac{1}{\sqrt{n}} (\Phi \Phi^+)_{lk}, \frac{1}{\sqrt{n}} (\Theta \Theta^+)_{kl} \right) \\ &\quad \times \exp \left\{ \sum_{k=1}^n \varphi_k^+ Z \varphi_k + \sum_{k=1}^n \vartheta_k^+ Z^* \vartheta_k \right\} d\Phi d\Theta \\ &= \int \exp \left\{ \sum_{k=1}^n \varphi_k^+ Z \varphi_k + \sum_{k=1}^n \vartheta_k^+ Z^* \vartheta_k \right. \\ &\quad \left. + \sum_{k,l=1}^n \log \psi \left(\frac{1}{\sqrt{n}} (\Phi \Phi^+)_{lk}, \frac{1}{\sqrt{n}} (\Theta \Theta^+)_{kl} \right) \right\} d\Phi d\Theta. \end{aligned}$$

Expansion of $\log \Phi$ into series gives us

$$\begin{aligned} f_m &= \int \exp \left\{ \sum_{k=1}^n \varphi_k^+ Z \varphi_k + \sum_{k=1}^n \vartheta_k^+ Z^* \vartheta_k \right. \\ &\quad \left. + \sum_{k,l=1}^n \sum_{p,s=0}^m \frac{\kappa_{p,s}}{p!s!} \frac{1}{n^{(p+s)/2}} ((\Phi \Phi^+)_{lk})^p ((\Theta \Theta^+)_{kl})^s \right\} d\Phi d\Theta, \end{aligned} \quad (2.17)$$

with

$$\kappa_{p,s} = \left. \frac{\partial^{p+s}}{\partial^p t_1 \partial^s t_2} \log \psi(t_1, t_2) \right|_{t_1=t_2=0}.$$

In particular,

$$\begin{aligned} \kappa_{0,0} &= 0; \\ \kappa_{1,0} &= \overline{\kappa_{0,1}} = \mathbf{E}\{x_{11}\} = 0; \\ \kappa_{2,0} &= \overline{\kappa_{0,2}} = \mathbf{E}\{x_{11}^2\} - \mathbf{E}^2\{x_{11}\} = 0; \\ \kappa_{1,1} &= \mathbf{E}\{|x_{11}|^2\} - |\mathbf{E}\{x_{11}\}|^2 = 1. \end{aligned} \quad (2.18)$$

Let us transform the terms in the exponent again

$$\begin{aligned}
& \sum_{k,l=1}^n ((\Phi\Phi^+)_{lk})^p ((\Theta\Theta^+)_{kl})^s \\
&= \sum_{k,l=1}^n \left(\sum_{j=1}^m \phi_{lj} \phi_{kj}^* \right)^p \left(\sum_{j=1}^m \theta_{kj} \theta_{lj}^* \right)^s = p!s! \sum_{k,l=1}^n \sum_{\substack{\alpha \in \mathcal{I}_{m,p} \\ \beta \in \mathcal{I}_{m,s}}} \prod_{q=1}^p \phi_{l\alpha_q} \phi_{k\alpha_q}^* \prod_{r=1}^s \theta_{k\beta_r} \theta_{l\beta_r}^* \\
&= (-1)^{p^2} p!s! \sum_{k,l=1}^n \sum_{\substack{\alpha \in \mathcal{I}_{m,p} \\ \beta \in \mathcal{I}_{m,s}}} \prod_{r=s}^1 \theta_{k\beta_r} \prod_{q=p}^1 \phi_{k\alpha_q}^* \prod_{q=1}^p \phi_{l\alpha_q} \prod_{r=1}^s \theta_{l\beta_r}^* \\
&= p!s! \sum_{\substack{\alpha \in \mathcal{I}_{m,p} \\ \beta \in \mathcal{I}_{m,s}}} \left(\sum_{k=1}^n (-1)^p \prod_{r=s}^1 \theta_{k\beta_r} \prod_{q=p}^1 \phi_{k\alpha_q}^* \right) \left(\sum_{k=1}^n \prod_{q=1}^p \phi_{k\alpha_q} \prod_{r=1}^s \theta_{k\beta_r}^* \right), \tag{2.19}
\end{aligned}$$

where

$$\mathcal{I}_{m,p'} = \{ \alpha \in \mathbb{Z}^{p'} \mid 1 \leq \alpha_1 < \dots < \alpha_{p'} \leq m \} \tag{2.20}$$

At this point the Hubbard–Stratonovich transformation is applied. As it was mentioned before, the transformation is an employment of (2.12) or (2.13) in the reverse direction. It yields for even $p + s$

$$\begin{aligned}
& \exp \left\{ \kappa_{p,s} n^{-(p+s)/2} \left(\sum_{k=1}^n (-1)^p \prod_{r=s}^1 \theta_{k\beta_r} \prod_{q=p}^1 \phi_{k\alpha_q}^* \right) \left(\sum_{k=1}^n \prod_{q=1}^p \phi_{k\alpha_q} \prod_{r=1}^s \theta_{k\beta_r}^* \right) \right\} \\
&= \frac{n}{\pi} \int \exp \left\{ -n^{-\frac{p+s-2}{4}} \sum_{k=1}^n \tilde{y}_{\beta\alpha}^{(k,p,s)} q_{\alpha\beta}^{(p,s)} - n^{-\frac{p+s-2}{4}} \sum_{k=1}^n \bar{q}_{\alpha\beta}^{(p,s)} y_{\alpha\beta}^{(k,p,s)} - n \left| q_{\alpha\beta}^{(p,s)} \right|^2 \right\} \\
&\times d\bar{q}_{\alpha\beta}^{(p,s)} dq_{\alpha\beta}^{(p,s)}, \tag{2.21}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{y}_{\beta\alpha}^{(k,p,s)} &= \sqrt{\kappa_{p,s}} (-1)^p \prod_{r=s}^1 \theta_{k\beta_r} \prod_{q=p}^1 \phi_{k\alpha_q}^*; \\
y_{\alpha\beta}^{(k,p,s)} &= \sqrt{\kappa_{p,s}} \prod_{q=1}^p \phi_{k\alpha_q} \prod_{r=1}^s \theta_{k\beta_r}^*. \tag{2.22}
\end{aligned}$$

Here and below we take a branch of the square root such that its argument is in $[0, \pi)$. Similarly, for odd $p + s$ we have

$$\begin{aligned}
& \exp \left\{ \kappa_{p,s} n^{-(p+s)/2} \left(\sum_{k=1}^n (-1)^p \prod_{r=1}^s \theta_{k\beta_r} \prod_{q=1}^p \phi_{k\alpha_q}^* \right) \left(\sum_{k=1}^n \prod_{q=1}^p \phi_{k\alpha_q} \prod_{r=1}^s \theta_{k\beta_r}^* \right) \right\} \\
&= \int \exp \left\{ -n^{-\frac{p+s}{4}} \sum_{k=1}^n \tilde{y}_{\beta\alpha}^{(k,p,s)} \xi_{\alpha\beta}^{(p,s)} - n^{-\frac{p+s}{4}} \sum_{k=1}^n \left(\xi_{\alpha\beta}^{(p,s)} \right)^* y_{\alpha\beta}^{(k,p,s)} \right. \\
&\quad \left. - \left(\xi_{\alpha\beta}^{(p,s)} \right)^* \xi_{\alpha\beta}^{(p,s)} \right\} d \left(\xi_{\alpha\beta}^{(p,s)} \right)^* d\xi_{\alpha\beta}^{(p,s)}. \tag{2.23}
\end{aligned}$$

Then the combination of (2.17), (2.19), (2.21) and (2.23) gives us

$$f_m = \left(\frac{n}{\pi}\right)^{c_m} \int \prod_{k=1}^n j_k \prod_{\substack{p+s \text{ is odd} \\ 0 \leq p, s \leq m}} e^{-\text{tr} \Xi_{p,s}^+ \Xi_{p,s}} d\Xi_{p,s}^+ d\Xi_{p,s} \prod_{\substack{p+s \text{ is even} \\ 0 \leq p, s \leq m}} e^{-n \text{tr} Q_{p,s}^* Q_{p,s}} dQ_{p,s}^* dQ_{p,s} \quad (2.24)$$

where

$$j_k = \int \exp \left\{ b_{k,2} + n^{-1/2} b_{k,4} + n^{-3/4} \mathbf{p}_a^{(1)}(\Xi, \Phi, \Theta) + n^{-1} \mathbf{p}_c^{(1)}(\hat{Q}, \Phi, \Theta) \right\} \\ \times d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k, \quad (2.25)$$

$$b_{k,2} = - \left(\text{tr} \tilde{Y}_{k,1,1} Q_{1,1} + \text{tr} Q_{1,1}^* Y_{k,1,1} \right) + \varphi_k^+ Z \varphi_k + \vartheta_k^+ Z^* \vartheta_k, \\ b_{k,4} = - \sum_{p+s=4} \left(\text{tr} \tilde{Y}_{k,p,s} Q_{p,s} + \text{tr} Q_{p,s}^* Y_{k,p,s} \right), \quad (2.26)$$

$$\mathbf{p}_a^{(1)}(\Xi, \Phi, \Theta) = - \sum_{j=2}^m n^{-(j-2)/2} \sum_{p+s=2j-1} \left(\text{tr} \tilde{Y}_{k,p,s} \Xi_{p,s} + \text{tr} \Xi_{p,s}^+ Y_{k,p,s} \right), \\ \mathbf{p}_c^{(1)}(\hat{Q}, \Phi, \Theta) = - \sum_{j=3}^m n^{-(j-3)/2} \sum_{p+s=2j} \left(\text{tr} \tilde{Y}_{k,p,s} Q_{p,s} + \text{tr} Q_{p,s}^* Y_{k,p,s} \right).$$

In the formulas above $\Xi_{p,s}$, $Q_{p,s}$, $\tilde{Y}_{k,p,s}$ and $Y_{k,p,s}$ are matrices whose entries are $\xi_{\alpha\beta}^{(p,s)}$, $q_{\alpha\beta}^{(p,s)}$, $\tilde{y}_{\beta\alpha}^{(k,p,s)}$ and $y_{\alpha\beta}^{(k,p,s)}$ respectively. The rows and columns are indexed by elements of the set $\mathcal{I}_{m,p}$ for corresponding p (or s) in the lexicographical order. Note also that $\mathbf{p}_a^{(1)}$ and $\mathbf{p}_c^{(1)}$ are the first degree homogeneous polynomials of the entries of Ξ and \hat{Q} respectively, where \hat{Q} contains all the $Q_{p,s}$ except Q_1 . One more thing we need is that all the monomials of $\mathbf{p}_a^{(1)}$ have odd degree w.r.t. φ_k and ϑ_k , and all the monomials of $\mathbf{p}_c^{(1)}$ have even degree w.r.t. φ_k and ϑ_k .

Fortunately, the integral in (2.24) over Φ and Θ factorizes. Therefore the integration can be performed over φ_k and ϑ_k separately for every k . Lemma 1 provides a corresponding result.

Lemma 1. *Let j_k be defined by (2.25). Then*

$$j_k = \det A + n^{-1/2} \tilde{h}(Q_2) + n^{-1} \mathbf{p}_c(\hat{Q}) + n^{-3/2} \mathbf{p}_a^{(2)}(\Xi, Q), \quad (2.27)$$

where A is defined in (2.4),

$$\tilde{h}(Q_2) = - \int \left(\text{tr} \tilde{Y}_{k,2,2} Q_2 + \text{tr} Q_2^* Y_{k,2,2} \right) e^{b_{k,2}} d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k, \quad (2.28)$$

$\mathbf{p}_c(\hat{Q})$ and $\mathbf{p}_a^{(2)}(\Xi, Q)$ are polynomials such that

(i) $\mathbf{p}_c(0) = 0$;

(ii) every monomial of $\mathbf{p}_a^{(2)}$ has at least second degree w.r.t. Ξ .

Proof. The integral j_k is computed by the expansion of the exponent into series. We start with

$$j_k = \int \left(1 + \sum_{1 \leq k \leq 4m/3} n^{-3k/4} (\mathbf{p}_a^{(1)}(\Xi, \Phi, \Theta))^k \right) e^{b_{k,2} + n^{-1/2} b_{k,4} + n^{-1} \mathbf{p}_c^{(1)}(\hat{Q}, \Phi, \Theta)} \\ \times d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k, \quad (2.29)$$

where the terms of degree higher than $4m$ w.r.t. φ_k and ϑ_k vanish, because the square of any anti-commuting variable is zero. The monomials of odd degree w.r.t. φ_k and ϑ_k also vanish after integration. Indeed, for every odd degree homogeneous polynomial \tilde{p} the expansion of $\tilde{p}(\varphi_k, \vartheta_k) e^{b_{k,2} + n^{-1/2}b_{k,4} + n^{-1}p_c(\hat{Q}, \Phi, \Theta)}$ into series gives us only odd degree terms. Whereas the number of Grassmann variables is even, there are no top degree monomials and the integral is zero. Thus (2.29) simplifies to

$$j_k = \int \left(1 + n^{-3/2}p_a^{(3)}(\Xi, \Phi, \Theta)\right) e^{b_{k,2} + n^{-1/2}b_{k,4} + n^{-1}p_c^{(1)}(\hat{Q}, \Phi, \Theta)} \times d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k, \quad (2.30)$$

where $p_a^{(3)}(\Xi, \Phi, \Theta)$ is a polynomial and its every monomial has degree at least 2 w.r.t. Ξ and at least 2 w.r.t. φ_k and ϑ_k . Note that

$$\int p_a^{(3)}(\Xi, \Phi, \Theta) e^{b_{k,2} + n^{-1/2}b_{k,4} + n^{-1}p_c^{(1)}(\hat{Q}, \Phi, \Theta)} d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k = p_a^{(2)}(\Xi, Q), \quad (2.31)$$

where $p_a^{(2)}(\Xi, Q)$ satisfies condition (ii). Substitution of (2.31) into (2.30) yields

$$j_k = \int e^{b_{k,2} + n^{-1/2}b_{k,4} + n^{-1}p_c^{(1)}(\hat{Q}, \Phi, \Theta)} d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k + n^{-3/2}p_a^{(2)}(\Xi, Q).$$

Further expansion implies

$$j_k = \int \left(1 + n^{-1/2}b_{k,4} + n^{-1}p_c^{(2)}(\hat{Q}, \Phi, \Theta)\right) e^{b_{k,2}} d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k + n^{-3/2}p_a^{(2)}(\Xi, Q),$$

where $p_c^{(2)}(\hat{Q}, \Phi, \Theta)$ is again a polynomial such that $p_c^{(2)}(0, \Phi, \Theta) = 0$. Similarly to above we obtain

$$j_k = \int \left(1 + n^{-1/2}b_{k,4}\right) e^{b_{k,2}} d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k + n^{-1}p_c(\hat{Q}) + n^{-3/2}p_a^{(2)}(\Xi, Q),$$

where $p_c(\hat{Q})$ satisfies condition (i).

Let us consider the expression (2.26) for $b_{k,4}$ in more detail. Every term in (2.26) with $(p, s) \neq (2, 2)$ has different numbers of “non-conjugate” Grassmann variables (without “+” superscript) and “conjugates” (with “+” superscript). But every term of $b_{k,2}$ has equal number of “non-conjugate” and “conjugate” Grassmann variables. The same is true for the expansion of $e^{b_{k,2}}$ and for top degree monomial of φ_k and ϑ_k . Hence for $(p, s) \neq (2, 2)$, $p + s = 4$

$$\int \left(\text{tr } \tilde{Y}_{k,p,s} Q_{p,s} + \text{tr } Q_{p,s}^* Y_{k,p,s}\right) e^{b_{k,2}} d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k = 0.$$

Therefore

$$j_k = \int \left(1 - n^{-1/2} \left(\text{tr } \tilde{Y}_{k,2,2} Q_2 + \text{tr } Q_2^* Y_{k,2,2}\right)\right) e^{b_{k,2}} d\varphi_k^+ d\varphi_k d\vartheta_k^+ d\vartheta_k + n^{-1}p_c(\hat{Q}) + n^{-3/2}p_a^{(2)}(\Xi, Q). \quad (2.32)$$

Recalling the definition of $y_{\alpha\beta}^{(k,p,s)}$ (2.22) and the values of $\kappa_{p,s}$ (2.18), one can render $b_{k,2}$ in the form

$$b_{k,2} = -\rho_k^+ A \rho_k, \quad (2.33)$$

where A is defined in (2.4) and ρ_k is defined in (2.16). Then (2.32) and (2.13) imply the assertion of the lemma. \square

A substitution of (2.27) into (2.24) gives us

$$\begin{aligned} f_m &= \left(\frac{n}{\pi}\right)^{c_m} \int (h(\mathbf{Q}) + n^{-3/2} \mathbf{p}_a^{(2)}(\Xi, \mathbf{Q}))^n \prod_{\substack{p+s \text{ is odd} \\ 0 \leq p, s \leq m}} e^{-\text{tr} \Xi_{p,s}^+ \Xi_{p,s}} d\Xi_{p,s}^+ d\Xi_{p,s} \\ &\quad \times \prod_{\substack{p+s \text{ is even} \\ 0 \leq p, s \leq m}} e^{-n \text{tr} Q_{p,s}^* Q_{p,s}} dQ_{p,s}^* dQ_{p,s}, \end{aligned}$$

where $h(\mathbf{Q})$ is defined in (2.3). Further

$$(h(\mathbf{Q}) + n^{-3/2} \mathbf{p}_a^{(2)}(\Xi, \mathbf{Q}))^n = \sum_{k=0}^{c_m} \binom{n}{k} n^{-3k/2} h(\mathbf{Q})^{n-k} (\mathbf{p}_a^{(2)}(\Xi, \mathbf{Q}))^k$$

because there are $2c_m$ anti-commuting variables and every monomial of $\mathbf{p}_a^{(2)}$ has at least second degree w.r.t. Ξ . Hence,

$$\begin{aligned} f_m &= \left(\frac{n}{\pi}\right)^{c_m} \int (h(\mathbf{Q})^{c_m} + n^{-1/2} \mathbf{p}_a^{(3)}(\Xi, \mathbf{Q})) \prod_{\substack{p+s \text{ is odd} \\ 0 \leq p, s \leq m}} e^{-\text{tr} \Xi_{p,s}^+ \Xi_{p,s}} d\Xi_{p,s}^+ d\Xi_{p,s} \\ &\quad \times e^{nf(\mathbf{Q}) - c_m \log h(\mathbf{Q})} d\mathbf{Q}, \end{aligned} \tag{2.34}$$

where $\mathbf{p}_a^{(3)}$ is a polynomial and $f(\mathbf{Q})$ is defined in (2.2). Taking into account (2.13) and the definition of an integral over anti-commuting variables, one can perform the integration over Ξ in (2.34) and obtain (2.1).

3 Asymptotic analysis

The goal of the section is to investigate the asymptotic behavior of the integral representation (2.5). To this end, the steepest descent method is applied. As usual, the hardest step is to choose stationary points of $f(\mathbf{Q})$ and a N -dimensional (real) manifold $M_* \subset \mathbb{C}^N$ such that for any chosen stationary point $\mathbf{Q}_* \in M_*$

$$\Re f(\mathbf{Q}) < \Re f(\mathbf{Q}_*), \quad \forall \mathbf{Q} \in M_*, \mathbf{Q} \text{ is not chosen.}$$

Note that N is equal to the number of real variables of the integration, i.e. in our case $N = 2^{2m}$.

The present proof proceeds a slightly different but rather standard scheme for the case when function $f(\mathbf{Q})$ has the form

$$f(\mathbf{Q}) = f_0(\mathbf{Q}) + n^{-1/2} f_r(\mathbf{Q}),$$

where $f_0(\mathbf{Q})$ does not depend on n , whereas $f_r(\mathbf{Q})$ may depend on n . We choose stationary points of $f_0(\mathbf{Q})$ of the form $Q_1 = U \lambda_0 V^*$, $\hat{\mathbf{Q}} = 0$, where λ_0 is a fixed real number and U and V vary in $U(m)$. Then the steepest descent method is applied to the integral over Λ and $\hat{\mathbf{Q}}$. In the process U and V are being considered as parameters and all the estimates are uniform in U and V . As soon as the domain of integration is restricted by a small neighborhood we recall about the integration over U and V . After several changes of the variables the integral is finally computed.

We start with the analysis of f_0 .

Lemma 2. Let the function $f_0: \mathbb{R}^{22m} \rightarrow [-\infty, +\infty)$ be defined by (2.6). Then $f_0(\Lambda, \hat{Q})$ attains its global maximum value only at the point

$$\lambda_1 = \cdots = \lambda_m = \lambda_0, \quad \hat{Q} = 0,$$

where $\lambda_0 = \sqrt{1 - |z_0|^2}$. Moreover, the matrix of second order derivatives of f_0 w.r.t. Λ and \hat{Q} at this point is negative definite.

Proof. It is evident from (2.6) and (2.7) that $f_0(\Lambda, \hat{Q})$ has the form

$$f_0(\Lambda, \hat{Q}) = \sum_{j=1}^m f_*(\lambda_j) - \sum_{(p,s) \neq (1,1)} \text{tr} Q_{p,s}^* Q_{p,s}, \quad (3.1)$$

where

$$f_*(\lambda) = -\lambda^2 + \log(|z_0|^2 + \lambda^2).$$

Since $f'_*(\lambda) = 0$ iff $\lambda = \lambda_0$ and $\lim_{\lambda \rightarrow \infty} f_*(\lambda) = -\infty$, $f_*(\lambda)$ attains its global maximum value only at $\lambda = \lambda_0$. Furthermore, $f''_*(\lambda_0) = -4\lambda_0^2$. These facts and (3.1) immediately imply the assertion of the lemma. \square

As in the previous section we consider first the Gaussian case and then the general case.

3.1 Gaussian case

Now we proceed to the integral estimates. In a standard way the integration domain in (2.10) can be restricted as follows

$$f_m = Cn^{m^2} \int_{\Sigma_r} \Delta^2(\Lambda^2) \prod_{j=1}^m \lambda_j \times e^{nf(U\Lambda V^*)} d\mu(U) d\mu(V) d\Lambda + O(e^{-nr/2}),$$

where

$$\Sigma_r = \{(\Lambda, U, V) \mid \|\Lambda\| \leq r\}.$$

The next step is to restrict the integration domain by

$$\Omega_n = \left\{ (\Lambda, U, V) \mid \|\Lambda - \Lambda_0\| \leq \frac{\log n}{\sqrt{n}} \right\}, \quad (3.2)$$

where $\Lambda_0 = \lambda_0 I$, I is a unit matrix. To this end we need the estimate of $\Re f$ given by the following lemmas.

Lemma 3. Let $\tilde{\Lambda}$ be a $m \times m$ diagonal matrix such that $\|\tilde{\Lambda}\| \leq \log n$. Then uniformly in U and V

$$\begin{aligned} f(U(\Lambda_0 + n^{-1/2}\tilde{\Lambda})V^*) &= -m\lambda_0^2 + n^{-1/2} \text{tr}(\bar{z}_0 \mathcal{Z} + z_0 \mathcal{Z}^*) + n^{-1} \text{tr} \mathcal{Z}_U \mathcal{Z}_V^* \\ &\quad - n^{-1} \text{tr}(2\lambda_0 \tilde{\Lambda} + \bar{z}_0 \mathcal{Z}_U + z_0 \mathcal{Z}_V^*)^2 / 2 + O(n^{-3/2} \log^3 n) \end{aligned} \quad (3.3)$$

where

$$\mathcal{Z}_B = B^* \mathcal{Z} B. \quad (3.4)$$

Proof. If $Q_1 = U(\Lambda_0 + n^{-1/2}\tilde{\Lambda})V^*$ then A has the form

$$A = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \left(A_0 + \frac{1}{\sqrt{n}} A_1 \right) \begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix},$$

where

$$A_0 = \begin{pmatrix} -Z_0 & \Lambda_0 \\ -\Lambda_0 & -Z_0^* \end{pmatrix}, \quad A_1 = \begin{pmatrix} -Z_U & \tilde{\Lambda} \\ -\tilde{\Lambda} & -Z_V^* \end{pmatrix}. \quad (3.5)$$

Taking into account that

$$\det A_0 = \left[\det \begin{pmatrix} -z_0 & \lambda_0 \\ -\lambda_0 & -\bar{z}_0 \end{pmatrix} \right]^m = 1,$$

one gets

$$\begin{aligned} \log \det A &= \log \det A_0^{-1} A = \operatorname{tr} \log(1 + n^{-1/2} A_0^{-1} A_1) \\ &= \frac{1}{\sqrt{n}} \operatorname{tr} A_0^{-1} A_1 - \frac{1}{2n} \operatorname{tr} (A_0^{-1} A_1)^2 + O\left(\frac{\log^3 n}{\sqrt{n^3}}\right) \end{aligned} \quad (3.6)$$

uniformly in U and V . Moreover,

$$A_0^{-1} A_1 = \begin{pmatrix} \bar{z}_0 Z_U + \lambda_0 \tilde{\Lambda} & -\bar{z}_0 \tilde{\Lambda} + \lambda_0 Z_V^* \\ -\lambda_0 Z_U + z_0 \tilde{\Lambda} & \lambda_0 \tilde{\Lambda} + z_0 Z_V^* \end{pmatrix}. \quad (3.7)$$

Combining (3.6), (3.7) and (2.11), we get

$$\begin{aligned} f(U(\Lambda_0 + n^{-1/2}\tilde{\Lambda})V^*) &= \operatorname{tr} \left[-\Lambda_0^2 - 2n^{-1/2}\lambda_0\tilde{\Lambda} - n^{-1}\tilde{\Lambda}^2 + n^{-1/2}(2\lambda_0\tilde{\Lambda} + \bar{z}_0 Z_U + z_0 Z_V^*) \right. \\ &\quad \left. - n^{-1}\{(\lambda_0^2 - |z_0|^2)\tilde{\Lambda}^2 + 2\bar{z}_0\lambda_0 Z_U \tilde{\Lambda} + 2z_0\lambda_0 Z_V^* \tilde{\Lambda} \right. \\ &\quad \left. + \frac{1}{2}(\bar{z}_0 Z_U + z_0 Z_V^*)^2 - Z_U Z_V^* \right] + O(n^{-3/2} \log^3 n). \end{aligned}$$

The last expansion yields (3.3). □

Lemma 4. *Let $\tilde{f}(Q_1) = f(Q_1) - f(\Lambda_0)$. Then for sufficiently large n*

$$\max_{\frac{\log n}{\sqrt{n}} \leq \|\Lambda - \Lambda_0\| \leq r} \Re \tilde{f}(U\Lambda V^*) \leq -C \frac{\log^2 n}{n}$$

uniformly in U and V .

Proof. First let us check that the first and the second derivatives of f_r are bounded in the δ -neighborhood of Λ_0 , where f_r is defined in (2.8) and δ is n -independent. Indeed, since h and h_0 are polynomials and $h \rightrightarrows h_0$ on compacts

$$\left| \frac{1}{\sqrt{n}} \frac{\partial \Re f_r}{\partial \lambda_j} \right| \leq \left| \frac{1}{\sqrt{n}} \frac{\partial f_r}{\partial \lambda_j} \right| = \left| \frac{\partial(f - f_0)}{\partial \lambda_j} \right| = \left| \frac{\partial(\log h - \log h_0)}{\partial \lambda_j} \right| \leq \left| \frac{1}{h_0} \cdot \frac{\partial h_0}{\partial \lambda_j} - \frac{1}{h} \cdot \frac{\partial h}{\partial \lambda_j} \right| \leq \frac{C}{\sqrt{n}}.$$

For every diagonal matrix $E = \operatorname{diag}\{e_j\}$ let $v(E)$ denote a vector with components e_j . Then for every diagonal matrix E of unit norm and for $\frac{\log n}{\sqrt{n}} \leq t \leq \delta$ we have

$$\begin{aligned} \frac{d}{dt} \Re \tilde{f}(U(\Lambda_0 + tE)V^*) &= \langle \nabla_{\Lambda} f_0(U(\Lambda_0 + tE)V^*), v(E) \rangle \\ &\quad + n^{-1/2} \langle \nabla_{\Lambda} \Re f_r(U(\Lambda_0 + tE)V^*), v(E) \rangle \\ &= \langle \nabla_{\Lambda} f_0(\Lambda_0 + tE), v(E) \rangle + O(n^{-1/2}), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is a standard scalar product. Expanding the scalar product by the Taylor formula and considering that $\nabla_{\Lambda} f_0(\Lambda_0) = 0$, we obtain

$$\frac{d}{dt} \Re \tilde{f}(U(\Lambda_0 + tE)V^*) = t \langle f_0''(\Lambda_0)v(E), v(E) \rangle + r_1 + O(n^{-1/2}),$$

where f_0'' is a matrix of second order derivatives of f_0 w.r.t. Λ and $|r_1| \leq Ct^2$. $f_0''(\Lambda_0)$ is negative definite according to Lemma 2. Hence $\frac{d}{dt} \Re \tilde{f}(U(\Lambda_0 + tE)V^*)$ is negative and

$$\max_{\frac{\log n}{\sqrt{n}} \leq \|\Lambda - \Lambda_0\| \leq \delta} \Re \tilde{f}(U\Lambda V^*) = \max_{\|\Lambda - \Lambda_0\| = \frac{\log n}{\sqrt{n}}} \Re \tilde{f}(U\Lambda V^*) \leq f(U\Lambda_0 V^*) - C \frac{\log^2 n}{n} - f(\Lambda_0). \quad (3.8)$$

Notice that f_r is bounded from above uniformly in n . This fact and Lemma 2 imply that δ in (3.8) can be replaced by r

$$\max_{\frac{\log n}{\sqrt{n}} \leq \|\Lambda - \Lambda_0\| \leq r} \Re \tilde{f}(U\Lambda V^*) \leq f(U\Lambda_0 V^*) - f(\Lambda_0) - C \frac{\log^2 n}{n}.$$

It remains to deduce from Lemma 3 that $f(U\Lambda_0 V^*) - f(\Lambda_0) = O(n^{-1})$ uniformly in U and V . \square

Lemma 4 yields

$$\mathbf{f}_m = Cn^{m^2} e^{nf(\Lambda_0)} \left(\int_{\Omega_n} \Delta^2(\Lambda^2) \prod_{j=1}^m \lambda_j \times e^{n\tilde{f}(U\Lambda V^*)} d\mu(U) d\mu(V) d\Lambda + O(e^{-C_1 \log^2 n}) \right),$$

where Ω_n is defined in (3.2). Changing the variables $\Lambda = \Lambda_0 + \frac{1}{\sqrt{n}} \tilde{\Lambda}$ and expanding f according to Lemma 3 we obtain

$$\begin{aligned} \mathbf{f}_m &= Ck_n \int_{\sqrt{n}\Omega_n} \Delta^2(\tilde{\Lambda}) \exp \left\{ -\text{tr}(2\lambda_0 \tilde{\Lambda} + \bar{z}_0 \mathcal{Z}_U + z_0 \mathcal{Z}_V^*)^2 / 2 + \text{tr} \mathcal{Z}_U \mathcal{Z}_V^* \right\} \\ &\quad \times d\mu(U) d\mu(V) d\tilde{\Lambda} (1 + o(1)), \end{aligned} \quad (3.9)$$

where

$$k_n = n^{m^2/2} e^{-mn\lambda_0^2 + \sqrt{n} \text{tr}(\bar{z}_0 \mathcal{Z} + z_0 \mathcal{Z}^*)}. \quad (3.10)$$

Let us change the variables $V = WU$. Taking into account that the Haar measure is invariant w.r.t. shifts we get

$$\begin{aligned} \mathbf{f}_m &= Ck_n \int_{\mathbb{R}^m} \int_{U(m)} \int_{U(m)} \Delta^2(\tilde{\Lambda}) \exp \left\{ -\text{tr}(2\lambda_0 \tilde{\Lambda} + U^*(\bar{z}_0 \mathcal{Z} + z_0 \mathcal{Z}_W^*)U)^2 / 2 + \text{tr} \mathcal{Z} W^* \mathcal{Z}^* W \right\} \\ &\quad \times d\mu(U) d\mu(W) d\tilde{\Lambda} (1 + o(1)) \\ &= Ck_n \int_{\mathbb{R}^m} \int_{U(m)} \int_{U(m)} \Delta^2(\tilde{\Lambda}) \exp \left\{ -\text{tr}(2\lambda_0 U \tilde{\Lambda} U^* + (\bar{z}_0 \mathcal{Z} + z_0 \mathcal{Z}_W^*))^2 / 2 + \text{tr} \mathcal{Z} W^* \mathcal{Z}^* W \right\} \\ &\quad \times d\mu(U) d\mu(W) d\tilde{\Lambda} (1 + o(1)). \end{aligned}$$

The next step is to change the variables $H = U \tilde{\Lambda} U^*$. The Jacobian is $\frac{\prod_{j=1}^{m-1} j!}{(2\pi)^{m(m-1)/2}} \Delta^{-2}(\tilde{\Lambda})$ (see e.g. [24]). Thus

$$\begin{aligned} \mathbf{f}_m &= Ck_n \int_{\mathcal{H}_m} \int_{U(m)} \exp \left\{ -\text{tr}(2\lambda_0 H + (\bar{z}_0 \mathcal{Z} + z_0 \mathcal{Z}_W^*))^2 / 2 + \text{tr} \mathcal{Z} W^* \mathcal{Z}^* W \right\} \\ &\quad \times d\mu(W) dH (1 + o(1)), \end{aligned}$$

where \mathcal{H}_m is a space of hermitian $m \times m$ matrices and

$$dH = \prod_{j=1}^m d(H)_{jj} \prod_{j < k} d\Re(H)_{jk} d\Im(H)_{jk}.$$

The Gaussian integration over H implies

$$f_m = Ck_n \int_{U(m)} \exp\{\text{tr } \mathcal{Z}W^* \mathcal{Z}^*W\} d\mu(W)(1 + o(1)). \quad (3.11)$$

If $\mathcal{Z} = 0$, (3.11) immediately yields (1.11). Otherwise, for computing the integral over the unitary group, the following Harish-Chandra/Itsykson–Zuber formula is used

Proposition 2. *Let A and B be normal $d \times d$ matrices with distinct eigenvalues $\{a_j\}_{j=1}^d$ and $\{b_j\}_{j=1}^d$ respectively. Then*

$$\int_{U(d)} \exp\{z \text{tr } AU^*BU\} d\mu(U) = \left(\prod_{j=1}^{d-1} j! \right) \frac{\det\{\exp(za_j b_k)\}_{j,k=1}^d}{z^{(d^2-d)/2} \Delta(A) \Delta(B)},$$

where z is some constant, μ is a Haar measure, and $\Delta(A) = \prod_{j > k} (a_j - a_k)$.

For the proof see, e.g., [27, Appendix 5]. Applying the Harish-Chandra/Itsykson–Zuber formula to (3.11) we obtain

$$f_m = Ck_n \frac{\det\{e^{\zeta_j \bar{\zeta}_k}\}_{j,k=1}^m}{|\Delta(\mathcal{Z})|^2} (1 + o(1)),$$

which in combination with (2.9) yields the result of Theorem 1.

3.2 General case

In the general case the proof proceeds by the same scheme as in the Gaussian case. In this subsection we focus on the crucial distinctions from the Gaussian case and refine the corresponding assertions from previous subsection. Set

$$\|\hat{Q}\| = \sum_{\substack{p+s \text{ is even} \\ 0 \leq p, s \leq m \\ (p,s) \neq (1,1)}} \|Q_{p,s}\|.$$

The generalization of Lemma 3 is

Lemma 5. *Let $\|\tilde{\Lambda}\| + \|\hat{Q}\| \leq \log n$. Then uniformly in U and V*

$$\begin{aligned} & f(U(\Lambda_0 + n^{-1/2} \tilde{\Lambda})V^*, n^{-1/2} \hat{Q}) \\ &= -m\lambda_0^2 + n^{-1/2} \text{tr}(\bar{z}_0 \mathcal{Z} + z_0 \mathcal{Z}^*) - n^{-1} \text{tr}(2\lambda_0 \tilde{\Lambda} + \bar{z}_0 \mathcal{Z}_U + z_0 \mathcal{Z}_V^*)^2/2 \\ &+ n^{-1} \text{tr } \mathcal{Z}_U \mathcal{Z}_V^* + n^{-1} \lambda_0^2 \sqrt{\kappa_{2,2}} \text{tr}(\wedge^2 VU^*) \tilde{Q}_2 + n^{-1} \lambda_0^2 \sqrt{\kappa_{2,2}} \text{tr } \tilde{Q}_2^* (\wedge^2 UV^*) \\ &- n^{-1} \sum_{\substack{p+s \text{ is even} \\ 0 \leq p, s \leq m \\ (p,s) \neq (1,1)}} \text{tr } \tilde{Q}_{p,s}^* \tilde{Q}_{p,s} + O(n^{-3/2} \log^3 n), \end{aligned} \quad (3.12)$$

where \mathcal{Z}_B is defined in (3.4) and $\wedge^2 B$ is the second exterior power of a linear operator B (see [40] for definition and properties of an exterior power of a linear operator).

Proof. Differently from the Gaussian case f has additional terms of the form $\text{tr } Q_{p,s}^* Q_{p,s}$ and additional term $n^{-1/2} \tilde{h}(Q_2) + n^{-1} \mathbf{p}_c(\hat{Q})$ under the logarithm (cf. (2.2) and (2.11)), where \tilde{h} and \mathbf{p}_c are defined in the assertion of Lemma 1. The contribution of the terms $\text{tr } Q_{p,s}^* Q_{p,s}$ to the expansion (3.12) is evident. Furthermore, $n^{-1} \mathbf{p}_c(n^{-1/2} \hat{Q}) = O(n^{-3/2} \log^3 n)$ because \mathbf{p}_c is a polynomial with zero constant term. Hence, it remains to determine the contribution of the term $n^{-1/2} \tilde{h}(Q_2)$.

In order to simplify notations, let us omit index k in (2.28). Thus, now $\boldsymbol{\varphi}$ and $\boldsymbol{\vartheta}$ denote vectors

$$\begin{pmatrix} \phi_{k1} \\ \vdots \\ \phi_{km} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \theta_{k1} \\ \vdots \\ \theta_{km} \end{pmatrix}$$

respectively. Then (2.28) is written as

$$\tilde{h}(Q_2) = - \int \left(\text{tr } \tilde{Y}_{2,2} Q_2 + \text{tr } Q_2^* Y_{2,2} \right) e^{b_2} d\boldsymbol{\varphi}^+ d\boldsymbol{\varphi} d\boldsymbol{\vartheta}^+ d\boldsymbol{\vartheta},$$

where $\tilde{Y}_{2,2}$ and $Y_{2,2}$ are defined in (2.22) and b_2 has the form (2.33). Therefore

$$\begin{aligned} n^{-1/2} \tilde{h}(n^{-1/2} \tilde{Q}_2) &= n^{-1} \tilde{h}(\tilde{Q}_2) = - \frac{\sqrt{\kappa_{2,2}}}{n} \int d\boldsymbol{\varphi}^+ d\boldsymbol{\varphi} d\boldsymbol{\vartheta}^+ d\boldsymbol{\vartheta} e^{-\boldsymbol{\rho}^+ A \boldsymbol{\rho}} \\ &\quad \times \sum_{\alpha, \beta \in \mathcal{I}_{m,2}} \left(\theta_{k\beta_1} \theta_{k\beta_2} \phi_{k\alpha_1}^* \phi_{k\alpha_2}^* \tilde{q}_{\alpha\beta}^{(2)} + \tilde{q}_{\alpha\beta}^{(2)} \phi_{k\alpha_1} \phi_{k\alpha_2} \theta_{k\beta_1}^* \theta_{k\beta_2}^* \right), \end{aligned} \quad (3.13)$$

where $\boldsymbol{\rho}$ is defined in (2.16), $\mathcal{I}_{m,2}$ is defined in (2.20). Let us change the variables $\tilde{\boldsymbol{\phi}} = U^* \boldsymbol{\varphi}$, $\tilde{\boldsymbol{\phi}}^+ = \boldsymbol{\varphi}^+ U$, $\tilde{\boldsymbol{\theta}} = V^* \boldsymbol{\vartheta}$, $\tilde{\boldsymbol{\theta}}^+ = \boldsymbol{\vartheta}^+ V$. We have

$$\begin{aligned} \theta_{k\beta_1} \theta_{k\beta_2} \phi_{k\alpha_1}^* \phi_{k\alpha_2}^* &= (V \tilde{\boldsymbol{\theta}})_{\beta_1} (V \tilde{\boldsymbol{\theta}})_{\beta_2} (\tilde{\boldsymbol{\phi}}^+ U^*)_{\alpha_1} (\tilde{\boldsymbol{\phi}}^+ U^*)_{\alpha_2} \\ &= \sum_{\gamma_1, \gamma_2=1}^m \sum_{\delta_1, \delta_2=1}^m v_{\beta_1 \gamma_1} \tilde{\theta}_{k\gamma_1} v_{\beta_2 \gamma_2} \tilde{\theta}_{k\gamma_2} \tilde{\phi}_{k\delta_1}^* \bar{u}_{\alpha_1 \delta_1} \tilde{\phi}_{k\delta_2}^* \bar{u}_{\alpha_2 \delta_2} \\ &= \sum_{\gamma, \delta \in \mathcal{I}_{m,2}} (v_{\beta_1 \gamma_1} v_{\beta_2 \gamma_2} - v_{\beta_1 \gamma_2} v_{\beta_2 \gamma_1}) \tilde{\theta}_{k\gamma_1} \tilde{\theta}_{k\gamma_2} \tilde{\phi}_{k\delta_1}^* \tilde{\phi}_{k\delta_2}^* \\ &\quad \times (\bar{u}_{\alpha_1 \delta_1} \bar{u}_{\alpha_2 \delta_2} - \bar{u}_{\alpha_1 \delta_2} \bar{u}_{\alpha_2 \delta_1}) \\ &= \sum_{\gamma, \delta \in \mathcal{I}_{m,2}} (\wedge^2 V)_{\beta\gamma} \tilde{\theta}_{k\gamma_1} \tilde{\theta}_{k\gamma_2} \tilde{\phi}_{k\delta_1}^* \tilde{\phi}_{k\delta_2}^* (\wedge^2 U^*)_{\delta\alpha}, \end{aligned} \quad (3.14)$$

where $u_{jk} = (U)_{jk}$, $v_{jk} = (V)_{jk}$. Similarly

$$\phi_{k\alpha_1} \phi_{k\alpha_2} \theta_{k\beta_1}^* \theta_{k\beta_2}^* = \sum_{\gamma, \delta \in \mathcal{I}_{m,2}} (\wedge^2 U)_{\alpha\gamma} \tilde{\phi}_{k\gamma_1} \tilde{\phi}_{k\gamma_2} \tilde{\theta}_{k\delta_1}^* \tilde{\theta}_{k\delta_2}^* (\wedge^2 V^*)_{\delta\beta}. \quad (3.15)$$

Besides,

$$\boldsymbol{\rho}^+ A \boldsymbol{\rho} = \tilde{\boldsymbol{\rho}}^+ \tilde{A} \tilde{\boldsymbol{\rho}} = \tilde{\boldsymbol{\rho}}^+ A_0 \tilde{\boldsymbol{\rho}} + O(n^{-1/2} \log n), \quad (3.16)$$

where A_0 is defined in (3.5) and

$$\tilde{\boldsymbol{\rho}} = \begin{pmatrix} \tilde{\boldsymbol{\phi}} \\ \tilde{\boldsymbol{\theta}} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix} A \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} -U^* Z U & \Lambda \\ -\Lambda & -V^* Z^* V \end{pmatrix}.$$

The “differentials” change as follows

$$\begin{aligned} d\boldsymbol{\varphi} &= \det^{-1} U d\tilde{\boldsymbol{\phi}}, \\ d\boldsymbol{\varphi}^+ &= \det^{-1} U^* d\tilde{\boldsymbol{\phi}}^+ \end{aligned} \quad (3.17)$$

and for $d\boldsymbol{\theta}$ likewise. Eventually, substitution of (3.14)–(3.17) into (3.13) yields

$$\begin{aligned} n^{-1}\tilde{h}(\tilde{Q}_2) &= -\frac{\sqrt{\kappa_{2,2}}}{n} \sum_{\gamma, \delta \in \mathcal{I}_{m,2}} \int \left(\tilde{\theta}_{k\gamma_1} \tilde{\theta}_{k\gamma_2} \tilde{\phi}_{k\delta_1}^* \tilde{\phi}_{k\delta_2}^* \left((\wedge^2 U^*) \tilde{Q}_2 (\wedge^2 V) \right)_{\delta\gamma} \right. \\ &\quad \left. + \left((\wedge^2 V^*) \tilde{Q}_2^* (\wedge^2 U) \right)_{\delta\gamma} \tilde{\phi}_{k\gamma_1} \tilde{\phi}_{k\gamma_2} \tilde{\theta}_{k\delta_1}^* \tilde{\theta}_{k\delta_2}^* \right) \\ &\quad \times e^{-\tilde{\rho}^+ A_0 \tilde{\rho}} d\tilde{\boldsymbol{\phi}}^+ d\tilde{\boldsymbol{\phi}} d\tilde{\boldsymbol{\theta}}^* d\tilde{\boldsymbol{\theta}} + O(n^{-3/2} \log^3 n) \end{aligned} \quad (3.18)$$

uniformly in U and V . Due to the structure of A_0 the integration can be performed over $\tilde{\phi}_{kj}$, $\tilde{\theta}_{kj}$ separately for every j . Notice that

$$\int \tau \exp \{ -\boldsymbol{\nu}^+ A' \boldsymbol{\nu} \} d\tilde{\phi}_{kj}^* d\tilde{\phi}_{kj} d\tilde{\theta}_{kj}^* d\tilde{\theta}_{kj} = 0,$$

where τ is either $\tilde{\phi}_{kj}^*$, $\tilde{\phi}_{kj}$, $\tilde{\theta}_{kj}^*$ or $\tilde{\theta}_{kj}$ and

$$\boldsymbol{\nu} = \begin{pmatrix} \tilde{\phi}_{kj} \\ \tilde{\theta}_{kj} \end{pmatrix}, \quad A' = \begin{pmatrix} -z_0 & \lambda_0 \\ -\lambda_0 & -\bar{z}_0 \end{pmatrix}.$$

Hence the terms with $\gamma \neq \delta$ in (3.18) are zeros. Furthermore, expanding the exponent into series, one can observe that

$$\int \tilde{\theta}_{kj} \tilde{\phi}_{kj}^* e^{-\boldsymbol{\nu}^+ A' \boldsymbol{\nu}} d\tilde{\phi}_{kj}^* d\tilde{\phi}_{kj} d\tilde{\theta}_{kj}^* d\tilde{\theta}_{kj} = - \int \tilde{\phi}_{kj} \tilde{\theta}_{kj}^* e^{-\boldsymbol{\nu}^+ A' \boldsymbol{\nu}} d\tilde{\phi}_{kj}^* d\tilde{\phi}_{kj} d\tilde{\theta}_{kj}^* d\tilde{\theta}_{kj} = \lambda_0.$$

It implies

$$\begin{aligned} n^{-1}\tilde{h}(\tilde{Q}_2) &= n^{-1} \lambda_0^2 \sqrt{\kappa_{2,2}} (\text{tr}(\wedge^2 U^*) \tilde{Q}_2 (\wedge^2 V) + \text{tr}(\wedge^2 V^*) \tilde{Q}_2^* (\wedge^2 U)) + o(n^{-1}) \\ &= n^{-1} \lambda_0^2 \sqrt{\kappa_{2,2}} (\text{tr}(\wedge^2 V U^*) \tilde{Q}_2 + \text{tr} \tilde{Q}_2^* (\wedge^2 U V^*)) + O(n^{-3/2} \log^3 n). \end{aligned}$$

The above relation completes the proof of (3.12). \square

An analog of Lemma 4 is

Lemma 6. *Let $\tilde{f}(\mathbf{Q}) = f(\mathbf{Q}) - f(\Lambda_0, 0)$. Then for sufficiently large n*

$$\max_{\substack{\log n \\ \sqrt{n}} \leq \|\Lambda - \Lambda_0\| + \|\hat{\mathbf{Q}}\| \leq r} \Re \tilde{f}(U \Lambda V^*, \hat{\mathbf{Q}}) \leq -C \frac{\log^2 n}{n}$$

uniformly in U and V .

The proof needs only cosmetic changes because of additional variables \hat{Q} . Following the proof in the Gaussian case one can see that (3.9) transforms into

$$\begin{aligned} f_m = Ck_n \int_{\sqrt{n}\Omega_n} \Delta^2(\tilde{\Lambda}) \exp \left\{ -\operatorname{tr}(2\lambda_0\tilde{\Lambda} + \bar{z}_0\mathcal{Z}_U + z_0\mathcal{Z}_V^*)^2/2 + \operatorname{tr} \mathcal{Z}_U\mathcal{Z}_V^* \right. \\ \left. + \lambda_0^2\sqrt{\kappa_{2,2}} \operatorname{tr}(\wedge^2 VU^*)\tilde{Q}_2 + \lambda_0^2\sqrt{\kappa_{2,2}} \operatorname{tr} \tilde{Q}_2^*(\wedge^2 UV^*) \right. \\ \left. - \sum_{\substack{p+s \text{ is even} \\ 0 \leq p,s \leq m \\ (p,s) \neq (1,1)}} \operatorname{tr} \tilde{Q}_{p,s}^* \tilde{Q}_{p,s} \right\} d\mu(U)d\mu(V)d\tilde{\Lambda}d\hat{Q}(1 + o(1)), \end{aligned}$$

where k_n is defined in (3.10). The Gaussian integration over \hat{Q} yields

$$\begin{aligned} f_m = Ck_n \exp \left\{ \frac{m^2 - m}{2} \lambda_0^4 \kappa_{2,2} \right\} \int \Delta^2(\tilde{\Lambda}) \exp \left\{ -\operatorname{tr}(2\lambda_0\tilde{\Lambda} + \bar{z}_0\mathcal{Z}_U + z_0\mathcal{Z}_V^*)^2/2 + \operatorname{tr} \mathcal{Z}_U\mathcal{Z}_V^* \right\} \\ \times d\mu(U)d\mu(V)d\tilde{\Lambda}(1 + o(1)). \end{aligned}$$

The last formula shows that there are no differences in further proof up to a high moments independent factor $\exp \left\{ \frac{m^2 - m}{2} \lambda_0^4 \kappa_{2,2} \right\}$.

References

- [1] G. Akemann and E. Kanzieper. Integrable structure of Ginibre's ensemble of real random matrices and a Pfaffian integration theorem. *J. Stat. Phys.*, 129(5-6):1159–1231, 2007.
- [2] G. Akemann and G. Vernizzi. Characteristic polynomials of complex random matrix models. *Nucl. Phys. B*, 660(3):532–556, 2003.
- [3] F. A. Berezin. *Introduction to superanalysis*. Number 9 in Math. Phys. Appl. Math. D. Reidel Publishing Co., Dordrecht, 1987. Edited and with a foreword by A. A. Kirillov. With an appendix by V. I. Ogievetsky. Translated from the Russian by J. Niederle and R. Kotecký. Translation edited by Dimitri Leites.
- [4] C. Bordenave and D. Chafaï. Around the circular law. *Probab. Surv.*, 9:1–89, 2012.
- [5] A. Borodin and E. Strahov. Averages of characteristic polynomials in random matrix theory. *Comm. Pure Appl. Math.*, 59(2):161–253, 2006.
- [6] E. Brézin and S. Hikami. Characteristic polynomials of random matrices. *Comm. Math. Phys.*, 214:111–135, 2000.
- [7] E. Brézin and S. Hikami. Characteristic polynomials of real symmetric random matrices. *Comm. Math. Phys.*, 223:363–382, 2001.
- [8] M. Disertori, M. Lohmann, and S. Sodin. The density of states of 1D random band matrices via a supersymmetric transfer operator. arXiv:1810.13150v1 [math.PR], 2018.
- [9] M. Disertori, T. Spencer, and M. R. Zirnbauer. Quasi-diffusion in a 3D supersymmetric hyperbolic sigma model. *Comm. Math. Phys.*, 300(2):435–486, 2010.
- [10] K. Efetov. *Supersymmetry in disorder and chaos*. Cambridge University Press, Cambridge, 1997.

- [11] K. B. Efetov. Supersymmetry and theory of disordered metals. *Adv. in Physics*, 32(1):53–127, 1983.
- [12] P. J. Forrester. Fluctuation formula for complex random matrices. *J. Phys. A*, 32(13):L159–L163, 1999.
- [13] Y. V. Fyodorov and Sommers H.-J. Random matrices close to Hermitian or unitary: overview of methods and results. *J. Phys. A*, 36(12):3303–3347, 2003.
- [14] Y. V. Fyodorov and B. A. Khoruzhenko. Systematic Analytical Approach to Correlation Functions of Resonances in Quantum Chaotic Scattering. *Phys. Rev. Lett.*, 83(1):65–68, 1999.
- [15] Y. V. Fyodorov and A. D. Mirlin. Localization in ensemble of sparse random matrices. *Phys. Rev. Lett.*, 67:2049–2052, 1991.
- [16] Y. V. Fyodorov and E. Strahov. An exact formula for general spectral correlation function of random Hermitian matrices. Random matrix theory. *J. Phys. A*, 36(12):3203–3214, 2003.
- [17] J. Ginibre. Statistical ensembles of complex, quaternion, and real matrices. *J. Mathematical Phys.*, 6:440–449, 1965.
- [18] V. L. Girko. The circular law. *Teor. Veroyatnost. i Primenen.*, 29(4):669–679, 1984.
- [19] V. L. Girko. The circular law: ten years later. *Random Oper. Stochastic Equations*, 2(3):235–276, 1994.
- [20] V. L. Girko. The strong circular law. Twenty years later. I. *Random Oper. Stochastic Equations*, 12(1):49–104, 2004.
- [21] V. L. Girko. The strong circular law. Twenty years later. II. *Random Oper. Stochastic Equations*, 12(3):255–312, 2004.
- [22] V. L. Girko. The circular law. Twenty years later. III. *Random Oper. Stochastic Equations*, 13(1):53–109, 2005.
- [23] T. Guhr. Supersymmetry. In G. Akemann, J. Baik, and P. D. Francesco, editors, *The Oxford Handbook of Random Matrix Theory*, chapter 7, pages 135–154. Oxford university press, 2015.
- [24] L. K. Hua. *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*. American Mathematical Society, Providence, RI, 1963.
- [25] P. Kopel. Linear Statistics of Non-Hermitian Matrices Matching the Real or Complex Ginibre Ensemble to Four Moments. preprint arXiv:1510.02987v1 [math.PR], 2015.
- [26] M. L. Mehta. *Random matrices and the statistical theory of energy levels*. Academic Press, New York–London, 1967.
- [27] M. L. Mehta. *Random Matrices*. Academic Press Inc., Boston, second edition, 1991.
- [28] A. D. Mirlin and Y. V. Fyodorov. Universality of level correlation function of sparse random matrices. *J. Phys. A*, 24:2273–2286, 1991.

- [29] C. Recher, M. Kieburg, T. Guhr, and M. R. Zirnbauer. Supersymmetry approach to Wishart correlation matrices: Exact results. *J. Stat. Phys.*, 148(6):981–998, 2012.
- [30] B. Rider and J. Silverstein. Gaussian fluctuations for non-Hermitian random matrix ensembles. *Ann. Probab.*, 34(6):2118–2143, 2006.
- [31] B. Rider and B. Virag. The noise in the circular law and the Gaussian free field. *Int. Math. Res. Not. IMRN*, 2:Art. ID rnm006, 33 pp, 2007.
- [32] M. Shcherbina and T. Shcherbina. Transfer matrix approach to 1d random band matrices: density of states. *J. Stat. Phys.*, 164(6):1233–1260, 2016.
- [33] M. Shcherbina and T. Shcherbina. Characteristic polynomials for 1D random band matrices from the localization side. *Comm. Math. Phys.*, 351(3):1009–1044, 2017.
- [34] M. Shcherbina and T. Shcherbina. Universality for 1d random band matrices: sigma-model approximation. *J. Stat. Phys.*, 172(2):627–664, 2018.
- [35] T. Shcherbina. On the correlation function of the characteristic polynomials of the Hermitian Wigner ensemble. *Comm. Math. Phys.*, 308:1–21, 2011.
- [36] T. Shcherbina. On the correlation functions of the characteristic polynomials of the Hermitian sample covariance matrices. *Probab. Theory Related Fields*, 156:449–482, 2013.
- [37] E. Strahov and Y. V. Fyodorov. Universal results for correlations of characteristic polynomials: Riemann-Hilbert approach. *Comm. Math. Phys.*, 241(2-3):343–382, 2003.
- [38] T. Tao and V. Vu. Random matrices: universality of ESDs and the circular law. *Ann. Probab.*, 38(5):2023–2065, 2010. With an appendix by Manjunath Krishnapur.
- [39] T. Tao and V. Vu. Random matrices: universality of local spectral statistics of non-Hermitian matrices. *Ann. Probab.*, 43(2):782–874, 2015.
- [40] E. B. Vinberg. *A Course in Algebra*. American Mathematical Society, Providence, RI, 2003.
- [41] C. Webb and M. D. Wong. On the moments of the characteristic polynomial of a Ginibre random matrix. *Proc. Lond. Math. Soc. (3)*, 118(5):1017–1056, 2019.