INVERSE SCATTERING TRANSFORM FOR THE TODA HIERARCHY WITH STEPLIKE FINITE-GAP BACKGROUNDS

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Abstract. We provide a rigorous treatment of the inverse scattering transform for the entire Toda hierarchy for solutions which are asymptotically close to (in general) different finite-gap solutions as $n \to \pm\infty$.

1. Introduction

The Toda lattice is one of the most prominent discrete integrable wave equations. In particular, it can be solved via the inverse scattering method. For the classical case, where the solution is asymptotically equal to the (same) constant solution, this is of course well understood and covered in several monographs (e.g.) [12], [36], or [33]. The corresponding long-time asymptotics were first computed by Novokshenov and Habibullin [29] and were later made rigorous by Kamvissis [16] under the additional assumption that no solitons are present (the case of solitons was recently added in [23]; see also the review [22]).

The inverse scattering transform for the entire Toda hierarchy in the case of a finite-gap background was solved only recently by us in [8] as a continuation of [31]. Similar results were obtained by Khanmamedov [21]. Long-time asymptotics for such solutions have been given by Kamvissis and Teschl for the case without solitons [18], [19], [20] and by Krüger and Teschl for the case with solitons [24] (for related trace formulas and conserved quantities see [27]).

In this respect it is important to mention that even the important case of a one-soliton solution on a finite-gap background has different spatial asymptotics as $n \to \pm\infty$ and hence is not covered by the above results (see [11], [34]). Hence this clearly raises the need to extend the results from [8] to the case of solutions which are asymptotically equal to (in general) different finite-gap solutions as $n \to \pm\infty$.

In fact, the simplest case, where the solution is asymptotically equal to two different constant solutions, has already attracted considerable interest in the past. The first to solve the corresponding Cauchy problem (in the case of rapid decay with respect to the background) seems to be Oba [30]. Moreover, the long-time asymptotics were considered in [1], [3], [4], [14], [15], [17], [37].

Our aim here is to fill this gap and to provide a treatment of the inverse scattering transform for the entire Toda hierarchy in the case of steplike quasi-periodic finite-gap backgrounds. Note that since we treat the entire Toda hierarchy, our results also cover the Kac–van Moerbeke hierarchy as a special case [28].

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Finally, we remark that the corresponding result for the Korteweg–de Vries equation is much more involved and was only recently solved by Egorova, Grunert, and Teschl [6] under some additional restrictions on the spectra of the background operators.

After introducing the Toda hierarchy in Section 2, we will first show that a solution will stay close to a given background solution in Section 3. This result implies that a short-range perturbation of a steplike finite-gap solution will stay short-range for all time, and it shows that the time-dependent scattering data satisfy the hypothesis necessary for the Gel'fand–Levitan–Marchenko theory [26]. This result constitutes the main technical ingredient for the inverse scattering transform. In Section 4 we review some necessary facts on quasi-periodic finite-gap solutions and in Section 5 we compute the time dependence of the scattering data and discuss its dynamics.

2. The Toda hierarchy

In this section we introduce the Toda hierarchy using the standard Lax formalism ([25]). We first review some basic facts from [2] (see also [13], [33]).

We will only consider bounded solutions and hence require

Hypothesis H. 2.1. *Suppose $a(t)$, $b(t)$ satisfy*

$$a(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \qquad b(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \qquad a(n,t) \neq 0, \qquad (n,t) \in \mathbb{Z} \times \mathbb{R},$$

and let $t \mapsto (a(t), b(t))$ be differentiable in $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$.

Associated with $a(t), b(t)$ is a Jacobi operator

$$H(t): \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \qquad f \mapsto \tau(t) f, \tag{2.1}$$

where

$$\tau(t) f(n) = a(n,t) f(n+1) + a(n-1,t) f(n-1) + b(n,t) f(n) \tag{2.2}$$

and $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. Moreover, choose constants $c_0 = 1$, c_j, $1 \le j \le r$, $c_{r+1} = 0$, set

$$\begin{aligned} g_j(n,t) &= \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_n, H(t)^\ell \delta_n \rangle, \\ h_j(n,t) &= 2a(n,t) \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_{n+1}, H(t)^\ell \delta_n \rangle + c_{j+1}, \end{aligned} \tag{2.3}$$

where $\langle \delta_m, A\delta_n \rangle$ denote the matrix elements of an operator A with respect to the standard basis, and consider the Lax operator

$$P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r} (2a(t) g_j(t) S^+ - h_j(t)) H(t)^{r-j} + g_{r+1}(t), \tag{2.4}$$

where $S^\pm f(n) = f(n \pm 1)$. Restricting to the two-dimensional nullspace

$$\mathrm{Ker}(\tau(t) - z), \quad z \in \mathbb{C},$$

of $\tau(t) - z$, we have the following representation of $P_{2r+2}(t)$:

$$P_{2r+2}(t)\Big|_{\mathrm{Ker}(\tau(t)-z)} = 2a(t) G_r(z,.,t) S^+ - H_{r+1}(z,.,t), \tag{2.5}$$

where $G_r(z,n,t)$ and $H_{r+1}(z,n,t)$ are monic polynomials in z of the type

$$G_r(z,n,t) = \sum_{j=0}^{r} z^j g_{r-j}(n,t),$$

$$H_{r+1}(z,n,t) = z^{r+1} + \sum_{j=0}^{r} z^j h_{r-j}(n,t) - g_{r+1}(n,t). \tag{2.6}$$

A straightforward computation shows that the Lax equation

$$\frac{d}{dt}H(t) - [P_{2r+2}(t), H(t)] = 0, \qquad t \in \mathbb{R}, \tag{2.7}$$

is equivalent to

$$\mathrm{TL}_r(a(t),b(t))_1 = \dot{a}(n,t) - a(n,t)\Big(g_{r+1}(n+1,t) - g_{r+1}(n,t)\Big) = 0,$$

$$\mathrm{TL}_r(a(t),b(t))_2 = \dot{b}(n,t) - \Big(h_{r+1}(n,t) - h_{r+1}(n-1,t)\Big) = 0, \tag{2.8}$$

where the dot denotes a derivative with respect to t. Varying $r \in \mathbb{N}_0$ yields the Toda hierarchy $\mathrm{TL}_r(a,b) = (\mathrm{TL}_r(a,b)_1, \mathrm{TL}_r(a,b)_2) = 0$. We will always consider r as a fixed, but arbitrary, value.

Finally, we recall that the Lax equation (2.7) implies existence of a unitary propagator $U_r(t,s)$ such that the family of operators $H(t)$, $t \in \mathbb{R}$, are unitarily equivalent, $H(t) = U_r(t,s)H(s)U_r(s,t)$.

3. The initial value problem

First of all we recall the basic existence and uniqueness theorem for the Toda hierarchy (see, e.g., [31], [32], or [33, Section 12.2]).

Theorem 3.1. *Suppose $(a_0,b_0) \in M = \ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$. Then there exists a unique integral curve $t \mapsto (a(t),b(t))$ in $C^\infty(\mathbb{R},M)$ of the Toda hierarchy, that is, $\mathrm{TL}_r(a(t),b(t)) = 0$, such that $(a(0),b(0)) = (a_0,b_0)$.*

In [31] it was shown that solutions which are asymptotically close to the constant solution at the initial time stay close for all time. Our first aim is to extend this result to include perturbations of quasi-periodic finite-gap solutions. In fact, we will even be a bit more general. Set

$$\|(a,b)\|_{w,p} = \begin{cases} \left(\sum_{n\in\mathbb{Z}} w(n)\Big(|a(n)|^p + |b(n)|^p\Big)\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{n\in\mathbb{Z}} w(n)\Big(|a(n)| + |b(n)|\Big), & p = \infty. \end{cases} \tag{3.1}$$

Then

Lemma 3.2. *Let $w(n) \ge 1$ be some weight with $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$ and fix some $1 \le p \le \infty$. Suppose $a(n,t)$, $b(n,t)$ and $a_\pm(n,t)$, $b_\pm(n,t)$ are arbitrary bounded solutions of the Toda hierarchy and abbreviate*

$$\bar{a}(n,t) = \begin{cases} a_+(n,t), & n \ge 0, \\ a_-(n,t), & n < 0, \end{cases} \qquad \bar{b}(n,t) = \begin{cases} b_+(n,t), & n \ge 0, \\ b_-(n,t), & n < 0. \end{cases} \tag{3.2}$$

Then, if

$$\|\big(a(t) - \bar{a}(t), b(t) - \bar{b}(t)\big)\|_{w,p} < \infty \tag{3.3}$$

holds for one $t = t_0 \in \mathbb{R}$, then it holds for all $t \in \mathbb{R}$.

Proof. Without loss of generality we assume that $t_0 = 0$. Let us consider the differential equation for the differences $\delta(n,t) = \big(a(n,t) - \bar{a}(n,t), b(n,t) - \bar{b}(n,t)\big)$ in the Banach space of pairs of bounded sequences $\delta = (\delta_1, \delta_2)$ for which the norm $\|\delta\|_{w,p}$ is finite. We claim that δ satisfies an inhomogeneous linear differential equation of the form

$$\dot{\delta}(t) = \sum_{|j|\le r+1} A_{r,j}(t)(S^+)^j \delta(t) + B_r(t)$$

(see e.g. [5] for the theory of ordinary differential equations in Banach spaces). Here $S^{\pm}(\delta_1(n,t), \delta_2(n,t)) = (\delta_1(n \pm 1, t), \delta_2(n \pm 1, t))$ are the shift operators,

$$A_{r,j}(n,t) = \begin{pmatrix} A^{11}_{r,j}(n,t) & A^{12}_{r,j}(n,t) \\ A^{21}_{r,j}(n,t) & A^{22}_{r,j}(n,t) \end{pmatrix},$$

are multiplication operators with bounded two by two matrix-valued sequences, and

$$B_r(n,t) = \begin{pmatrix} B_{r,1}(n,t) \\ B_{r,2}(n,t) \end{pmatrix}$$

is a vector in our Banach space with $B_{r,i}(n,t) = 0$ for $|n| > \lfloor \frac{r}{2} \rfloor + 1$. All entries of $A_{r,j}(t)$ and $B_r(t)$ are polynomials with respect to $(a(n+j,t), b(n+j,t))$, $(a_\pm(n+j,t), b_\pm(n+j,t))$, $|j| \le \lfloor \frac{r}{2} \rfloor + 1$. Moreover, by our assumption the shift operators are continuous,

$$\|S^{\pm}\| = \begin{cases} \sup_{n\in\mathbb{Z}} |\frac{w(n)}{w(n\pm 1)}|^{1/p}, & p \in [1,\infty), \\ \sup_{n\in\mathbb{Z}} |\frac{w(n)}{w(n\pm 1)}|, & p = \infty, \end{cases}$$

and same is true for the multiplication operators $A_{r,j}(t)$ whose norms depend only on the supremum of the entries by Hölder's inequality, that is, on the sup norms of $(a(t), b(t))$ and $(a_\pm(t), b_\pm(t))$. Finally, recall that by unitary equivalence of the operator family $H(t)$, respectively $H_\pm(t)$, we have a uniform bound of the sup norm $\sup_n(|a(n,t)| + |b(n,t)|) \le 2\|H(t)\| = 2\|H(0)\|$, respectively $\sup_n(|a_\pm(n,t)| + |b_\pm(n,t)|) \le 2\|H_\pm(t)\| = 2\|H_\pm(0)\|$. Consequently, there is a constant such that $\sum_{|j|\le r+1} \|A_{r,j}(t)\| \|(S^+)^j\| \le C_r$. Moreover, we will show below that the vector $B_r(t)$ has only finitely many nonzero entries and thus $\|B_r(t)\|_{w,p} \le D_r$, where the constant again depends only on the sup norms of $(a(t), b(t))$ and $(a_\pm(t), b_\pm(t))$. Hence

$$\|\delta(t)\|_{w,p} \le \|\delta(0)\|_{w,p} + \int_0^t \big(C_r \|\delta(s)\|_{w,p} + D_r\big)$$

and Gronwall's inequality implies

$$\|\delta(t)\|_{w,p} \le \|\delta(0)\|_{w,p} \mathrm{e}^{C_r t} + \frac{D_r}{C_r}(\mathrm{e}^{C_r t} - 1).$$

It remains to show existence of the above differential equation. This will follow once we show that $g_{r+1}(t) - \bar{g}_{r+1}(t)$ and $h_{r+1}(t) - \bar{h}_{r+1}(t)$ can be written as a linear combination of shifts of δ with the coefficients depending only on $(a(t), b(t))$ and $(a_\pm(t), b_\pm(t))$. The fact that $(\bar{a}, \bar{b})$ does not solve TL_r only affects finitely many terms and gives rise to the inhomogeneous term $B_r(t)$ which is nonzero only for a finite number of terms.

To see that $g_{r+1}(t) - \bar{g}_{r+1}(t)$ and $h_{r+1}(t) - \bar{h}_{r+1}(t)$ can be written as a linear combination of shifts of δ we can use induction on r. It suffices to consider the homogenous case where $c_j = 0$, $1 \le j \le r$, since all involved sums are finite. In this case [33, Lemma 6.4] shows that $g_j(n,t)$, $h_j(n,t)$ can be recursively computed from $g_0(n,t) = 1$, $h_0(n,t) = 0$ via

$$g_{j+1}(n,t) = \frac{1}{2}\big(h_j(n,t) + h_j(n-1,t)\big) + b(n,t) g_j(n,t),$$

$$h_{j+1}(n,t) = 2a(n,t)^2 \sum_{l=0}^{j} g_{j-l}(n,t) g_l(n+1,t) - \frac{1}{2}\sum_{l=0}^{j} h_{j-l}(n,t) h_l(n,t)$$

and similarly for $\bar{g}_j(n,t)$, $\bar{h}_j(n,t)$. Hence the claim follows.

Finally, observe that since $w(n) \ge 1$ this solution is bounded and hence coincides with the solution of the Toda equation from Theorem 3.1. $\square$

For closely related results we also refer to [35].

4. Quasi-periodic finite-gap solutions

As a preparation for our next section we first need to recall some facts on quasi-periodic finite-gap solutions (again see [2], [13], or [33]).

Let $H_q^{\pm}$ be two quasi-periodic finite-band Jacobi operators,[1]

$$H_q^{\pm}(t) f(n) = a_q^{\pm}(n,t) f(n+1) + a_q^{\pm}(n-1,t) f(n-1) + b_q^{\pm}(n,t) f(n) \tag{4.1}$$

in $\ell^2(\mathbb{Z})$ associated with the Riemann surface of the square root

$$P_{\pm}(z) = -\prod_{j=0}^{2g_{\pm}+1} \sqrt{z - E_j^{\pm}}, \qquad E_0^{\pm} < E_1^{\pm} < \cdots < E_{2g_{\pm}+1}^{\pm}, \tag{4.2}$$

where $g_{\pm} \in \mathbb{N}$ and $\sqrt{\cdot}$ is the standard root with branch cut along $(-\infty, 0)$. In fact, $H_q^{\pm}(t)$ are uniquely determined by fixing a Dirichlet divisor $\sum_{j=1}^{g^{\pm}} (\mu_j^{\pm}(t), \sigma_j^{\pm}(t))$, where $\mu_j^{\pm}(t) \in [E_{2j-1}^{\pm}, E_{2j}^{\pm}]$ and $\sigma_j^{\pm}(t) \in \{-1, 1\}$. The time evolution of the Dirichlet divisor is determined by the Dubrovin equations (cf. [33, (13.2)])and linearized by the Abel map (cf. [33, Sect. 13.2]). The spectra of $H_q^{\pm}(t)$ consist of $g_{\pm}+1$ bands

$$\sigma_{\pm} := \sigma(H_q^{\pm}(t)) = \bigcup_{j=0}^{g_{\pm}} [E_{2j}^{\pm}, E_{2j+1}^{\pm}]. \tag{4.3}$$

We will identify the set $\mathbb{C} \setminus \sigma(H_q^{\pm}(t))$ with the upper sheet of the Riemann surface. Associated with $H_q^{\pm}(t)$ are the Weyl solutions

$$\psi_q^{\pm}(z,n,t) \in \ell^2(\pm\mathbb{N}) \tag{4.4}$$

normalized such that $\psi_q^{\pm}(z,0,t) = 1$. We will use the convention that for $\lambda \in \sigma_{\pm}$ we set $\psi_q^{\pm}(\lambda,n,t) = \lim_{\epsilon\downarrow 0} \psi_q^{\pm}(\lambda + \mathrm{i}\epsilon)$. Then

$$\hat{\psi}_q^{\pm}(z,n,t) = \exp\big(\alpha_r^{\pm}(z,t)\big)\psi_q^{\pm}(z,n,t) \tag{4.5}$$

satisfies

$$H_q^{\pm}(t)\hat{\psi}_q^{\pm}(z,n,t) = z\hat{\psi}_q^{\pm}(z,n,t), \tag{4.6}$$

$$\frac{d}{dt}\hat{\psi}_q^{\pm}(z,n,t) = P_{q,2r+2}^{\pm}(t)\hat{\psi}_q^{\pm}(z,n,t), \tag{4.7}$$

[1]Everywhere in this paper the sub or super index "+" (resp. "−") refers to the background on the right (resp. left) half-axis.

where ([33], (13.47))

$$\alpha_r^\pm(z,t) = \int_0^t \big(2a_q^\pm(0,s)G_{q,r}^\pm(z,0,s)\psi_q^\pm(z,1,s) - H_{q,r+1}^\pm(z,0,s)\big)ds. \tag{4.8}$$

Note that the integrand in this last expression might have poles if z lies in one of the spectral gaps $[E_{2j-1}^\pm, E_{2j}^\pm]$. Hence one has to understand $\alpha_r^\pm(z,t)$ as a limit from $z \in \mathbb{C}\backslash\mathbb{R}$ for such values of z. Alternatively one can use the expression in terms of Riemann theta functions. Moreover, $\exp\big(\alpha_r^\pm(z,t)\big)$ has simple poles at $\mu_j^\pm(0)$ and simple zeros at $\mu_j^\pm(t)$. We refer to the discussion in [8] for further details.

5. Inverse scattering transform

Fix two quasi-periodic finite-gap solutions $a_q^\pm(n,t)$, $b_q^\pm(n,t)$ as in the previous section. Let $a(n,t)$, $b(n,t)$ be a solution of the Toda hierarchy satisfying

$$\sum_{n=0}^{\pm\infty}(1+|n|)\Big(|a(n,t)-a_q^\pm(n,t)| + |b(n,t)-b_q^\pm(n,t)|\Big) < \infty \tag{5.1}$$

for one (and hence for any) $t_0 \in \mathbb{R}$. In [10] (see also [7], [9], [38]) we have developed scattering theory for the Jacobi operator $H(t)$ associated with $a(n,t)$, $b(n,t)$. Jost solutions, transmission and reflection coefficients now depend on an additional parameter $t \in \mathbb{R}$. The essential spectrum of $H(t)$ is (absolutely) continuous and

$$\sigma(H(t)) \equiv \sigma(H), \quad \sigma_{ess}(H) = \sigma_+ \cup \sigma_-, \quad \sigma_p(H) = \{\lambda_k\}_{k=1}^p \subseteq \mathbb{R}\backslash\sigma_{ess}(H), \tag{5.2}$$

where $p \in \mathbb{N}$ is finite. We introduce the sets

$$\sigma^{(2)} := \sigma_+ \cap \sigma_-, \quad \sigma_\pm^{(1)} = \operatorname{clos}(\sigma_\pm \setminus \sigma^{(2)}), \quad \sigma := \sigma_+ \cup \sigma_-, \tag{5.3}$$

where σ is the (absolutely) continuous spectrum of $H(t)$ and $\sigma_+^{(1)} \cup \sigma_-^{(1)}$, $\sigma^{(2)}$ are the parts which are of multiplicity one, two, respectively.

The Jost solutions $\psi_\pm(z,n,t)$ are normalized such that

$$\psi_\pm(z,n,t) = \psi_q^\pm(z,n,t)\,(1+o(1)) \quad \text{as } n \to \pm\infty. \tag{5.4}$$

Transmission $T_\pm(\lambda,t)$ and reflection $R_\pm(\lambda,t)$ coefficients are defined via the scattering relations

$$T_\mp(\lambda,t)\psi_\pm(\lambda,n,t) = \overline{\psi_\mp(\lambda,n,t)} + R_\mp(\lambda,t)\psi_\mp(\lambda,n,t), \qquad \lambda \in \sigma_\mp, \tag{5.5}$$

which implies

$$T_\pm(\lambda,t) := \frac{W(\overline{\psi_\pm(\lambda,t)},\psi_\pm(\lambda,t))}{W(\psi_\mp(\lambda,t),\psi_\pm(\lambda,t))}, \quad R_\pm(\lambda,t) := -\frac{W(\psi_\mp(\lambda,t),\overline{\psi_\pm(\lambda,t)})}{W(\psi_\mp(\lambda,t),\,\psi_\pm(\lambda,t))}, \tag{5.6}$$

$\lambda \in \sigma_\pm$. Here $W_n(f,g) = a(n)(f(n)g(n+1) - f(n+1)g(n))$ denotes the usual Wronski determinant.

To define the norming constants we need to remove the poles of $\psi^\pm(z,n,t)$ by introducing

$$\tilde\psi^\pm(z,n,t) = \delta_\pm(z,t)\psi^\pm(z,n,t), \qquad \delta_\pm(z,t) := \prod_{\mu_j^\pm(t)\in M_\pm(t)} (z-\mu_j^\pm(t)), \tag{5.7}$$

where

$$M^\pm(t) = \{\mu_j^\pm(t)\,|\,\mu_j^\pm(t) \in \mathbb{R}\backslash\sigma_\pm \text{ is a pole of } \psi_q^\pm(z,1,t)\}. \tag{5.8}$$

The norming constants $\gamma_{\pm,k}(t)$ corresponding to $\lambda_k \in \sigma_p(H)$ are then given by

$$\gamma_{\pm,k}(t)^{-1} = \sum_{n\in\mathbb{Z}} |\tilde\psi_\pm(\lambda_k,n,t)|^2. \tag{5.9}$$

Lemma 5.1. *Let $(a(t),b(t))$ be a solution of the Toda hierarchy such that* (5.1) *holds. The functions*

$$\hat\psi_\pm(z,n,t) = \exp(\alpha_r^\pm(z,t))\psi_\pm(z,n,t) \tag{5.10}$$

satisfy

$$H(t)\hat\psi_\pm(z,n,t) = z\hat\psi_\pm(z,n,t), \qquad \frac{d}{dt}\hat\psi_\pm(z,n,t) = P_{2r+2}(t)\hat\psi_\pm(z,n,t). \tag{5.11}$$

Proof. We proceed as in [31, Theorem 3.2]. The Jost solutions $\psi_\pm(z,n,t)$ are continuously differentiable with respect to t by the same arguments as for z (compare [7, Theorem 4.2]), and the derivatives are equal to the derivatives of the Baker–Akhiezer functions as $n \to \pm\infty$.

For $z \in \mathbb{C}\backslash\sigma$, the solution $u_\pm(z,n,t)$ of (5.11) with initial condition $\psi_\pm(z,n,0) \in \ell^2_\pm(\mathbb{Z})$ remains square summable near $\pm\infty$ for all $t \in \mathbb{R}$ (see [32] or [33, Lemma 12.16]), that is, $u_\pm(z,n,t) = C_\pm(z,t)\psi_\pm(z,n,t)$. Letting $n \to \pm\infty$ we see $C_\pm(z,t) = 1$. The general result for all $z \in \mathbb{C}$ now follows from continuity. □

This implies

Theorem 5.2. *Let $(a(t),b(t))$ be a solution of the Toda hierarchy such that* (5.1) *holds. The time evolution for the scattering data is given by*

$$\begin{aligned}
T_\pm(\lambda,t) &= T_\pm(\lambda,0)\exp(\alpha_r^\mp(\lambda,t) - \overline{\alpha_r^\pm(\lambda,t)}),\\
R_\pm(\lambda,t) &= R_\pm(\lambda,0)\exp(\alpha_r^\pm(\lambda,t) - \overline{\alpha_r^\pm(\lambda,t)}),\\
\gamma_{\pm,k}(t) &= \gamma_{\pm,k}(0)\frac{\delta_\pm^2(\lambda_k,0)}{\delta_\pm^2(\lambda_k,t)}\exp(2\alpha_r^\pm(\lambda_k,t)), \qquad 1 \le k \le p.
\end{aligned} \tag{5.12}$$

Proof. The Wronskian of two solutions satisfying (5.11) does not depend on n or t (see [32], [33, Lemma 12.15]), hence

$$\begin{aligned}
T_\pm(\lambda,t) &= \frac{W(\overline{\psi_\pm(\lambda,t)},\psi_\pm(\lambda,t))}{W(\psi_\mp(\lambda,t),\psi_\pm(\lambda,t))} = \frac{\exp(\alpha_r^\mp(\lambda,t))}{\exp(\overline{\alpha_r^\pm(\lambda,t)})}\frac{W(\overline{\hat\psi_\pm(\lambda,t)},\hat\psi_\pm(\lambda,t))}{W(\hat\psi_\mp(\lambda,t),\hat\psi_\pm(\lambda,t))}\\
&= \exp(\alpha_r^\mp(\lambda,t) - \overline{\alpha_r^\pm(\lambda,t)})T_\pm(\lambda,0),\\
R_\pm(\lambda,t) &= -\frac{W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})}{W(\psi_\mp(\lambda),\psi_\pm(\lambda))} = -\frac{\exp(\alpha_r^\pm(\lambda,t))}{\exp(\overline{\alpha_r^\pm(\lambda,t)})}\frac{W(\hat\psi_\mp(\lambda),\overline{\hat\psi_\pm(\lambda)})}{W(\hat\psi_\mp(\lambda),\hat\psi_\pm(\lambda))}\\
&= \exp(\alpha_r^\pm(\lambda,t) - \overline{\alpha_r^\pm(\lambda,t)})R_\pm(\lambda,0).
\end{aligned}$$

The time dependence of $\gamma_{\pm,k}(t)$ follows from $\|U_r(t,0)\tilde\psi_\pm(\lambda_k,.,0)\| = \|\tilde\psi_\pm(\lambda_k,.,0)\|$. □

Remark 5.3. *Note that we have*

$$\exp\left(\alpha_r^\pm(\lambda,t) - \overline{\alpha_r^\pm(\lambda,t)}\right) = \exp\left(P_\pm(\lambda)\int_0^t \frac{G_{q,r}^\pm(\lambda,0,s)}{\prod_{j=1}^{g_\pm}(\lambda-\mu_j^\pm(s))}ds\right), \quad \lambda \in \sigma_\pm, \tag{5.13}$$

where $\mu_j^\pm(t)$ are the Dirichlet eigenvalues. Moreover, in case of the Toda lattice, where $r = 0$, we have $G^\pm_{q,0}(\lambda, n, t) = 1$ and $H^\pm_{q,1}(\lambda, n, t) = \lambda - b_q^\pm(n, t)$.

In summary, since Lemma 3.2 ensures that (5.1) remains valid for all t once it holds for the initial condition, we can compute $R_\pm(\lambda, 0)$ and $\gamma_{\pm,k}(0)$ from $(a(n, 0), b(n, 0))$ and then solve the Gel'fand–Levitan–Marchenko (GLM) equation to obtain the sequences $(a(n, t), b(n, t))$ as in [10]. More precisely, one needs to solve the GLM equation

$$K_\pm(n, m, t) + \sum_{l=n}^{\pm\infty} K_\pm(n, l, t) F_\pm(l, m, t) = \frac{\delta_n(m)}{K_\pm(n, n, t)}, \qquad \pm m \geq \pm n, \tag{5.14}$$

for $K_\pm(n, m, t)$, where according to Theorem 5.2 the kernel $F_\pm(m, n, t)$ is given by

Theorem 5.4. *The time dependence of the kernel of the Gel'fand–Levitan–Marchenko equation is given by*

$$\begin{aligned} F_\pm(m, n, t) = {} & \frac{1}{\pi} \mathrm{Re} \int_{\sigma_\pm} R_\pm(\lambda, 0) \hat\psi_q^\pm(\lambda, m, t) \hat\psi_q^\pm(\lambda, n, t) \frac{\prod_{j=1}^{g_\pm}(\lambda - \mu_j^\pm(0))}{P_\pm(\lambda)} d\lambda \\ & + \frac{1}{2\pi\mathrm{i}} \int_{\sigma_\mp^{(1)}} |T_\mp(\lambda, 0)|^2 \hat\psi_q^\pm(\lambda, m, t) \hat\psi_q^\pm(\lambda, n, t) \frac{\prod_{j=1}^{g_\pm}(\lambda - \mu_j^\pm(0))}{P_\mp(\lambda)} d\lambda \\ & + \sum_{k=1}^{p} \gamma_{\pm,k}(0) \breve\psi_q^\pm(\lambda_k, m, t) \breve\psi_q^\pm(\lambda_k, n, t), \end{aligned} \tag{5.15}$$

where $\breve\psi_q^\pm(z, m, t) = \delta_\pm(z, 0) \hat\psi_q^\pm(z, m, t)$.

Proof. The kernel $F_\pm(m, n, 0)$ is derived in [10, Theorem 4.1]. Observe that $\alpha_r^\mp(\lambda, t)$ are real valued on the set $\sigma_\mp^{(1)}$ and

$$\exp\left(\alpha_r^\pm(\lambda, t) + \overline{\alpha_r^\pm(\lambda, t)}\right) = \prod_{j=1}^{g_\pm} \frac{\lambda - \mu_j^\pm(t)}{\lambda - \mu_j^\pm(0)}, \quad \lambda \in \sigma_\pm,$$

then our result follows from (5.10) and Theorem 5.2. □

By [10] this equation is uniquely solvable and the solution of the Toda hierarchy can be obtained from either $K_+(n, m, t)$ or $K_-(n, m, t)$ by virtue of

$$\begin{aligned} a(n, t) &= a_q^+(n, t) \frac{K_+(n+1, n+1, t)}{K_+(n, n, t)} = a_q^-(n, t) \frac{K_-(n, n, t)}{K_-(n+1, n+1, t)}, \\ b(n, t) &= b_q^+(n, t) + a_q^+(n, t) \frac{K_+(n, n+1, t)}{K_+(n, n, t)} - a_q^+(n-1, t) \frac{K_+(n-1, n, t)}{K_+(n-1, n-1, t)}, \\ &= b_q^-(n, t) + a_q^-(n-1, t) \frac{K_-(n, n-1, t)}{K_-(n, n, t)} - a_q^-(n, t) \frac{K_-(n+1, n, t)}{K_-(n+1, n+1, t)}. \end{aligned} \tag{5.16}$$

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