SCATTERING THEORY WITH FINITE-GAP BACKGROUNDS: TRANSFORMATION OPERATORS AND CHARACTERISTIC PROPERTIES OF SCATTERING DATA

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ABSTRACT. We develop direct and inverse scattering theory for Jacobi operators (doubly infinite second order difference operators) with steplike coefficients which are asymptotically close to different finite-gap quasi-periodic coefficients on different sides. We give necessary and sufficient conditions for the scattering data in the case of perturbations with finite second (or higher) moment.

1. INTRODUCTION

The present paper is concerned with inverse scattering theory for Jacobi operators (doubly infinite second order difference operators) given by

$$(1.1)\quad Hf(n) = a(n-1)f(n-1) + b(n)f(n) + a(n)f(n+1), \quad a(n) > 0, \ b(n) \in \mathbb{R},$$

and acting on the Hilbert space $\ell^2(\mathbb{Z})$ of doubly infinite square summable sequences. We will consider this problem in the case where the coefficients of H are asymptotically close to the coefficients of two (in general different) finite-gap operators

$$(1.2)\quad H^\pm f(n) = a^\pm(n)f(n+1) + a^\pm(n-1)f(n-1) + b^\pm(n)f(n)$$

as $n \to \pm\infty$[1]. By a finite-gap operator we mean an operator associated with quasiperiodic sequences whose spectrum consists of finitely many bands. In particular, this covers the case of periodic or constant Jacobi operators.

Definition 1.1. *Let $q = 0, 1, 2, \dots$ be a nonnegative integer. We say that H has the q'th moment finite with respect to H^+, H^- if its coefficients satisfy*

$$(1.3)\quad \sum_{n=0}^{\pm\infty} |n|^q \Big(|a(n) - a^\pm(n)| + |b(n) - b^\pm(n)| \Big) < \infty.$$

The set of all Jacobi operators H satisfying (1.3) *will be denoted by $\mathcal{B}_q(H^+, H^-)$.*

Given two fixed limiting (or background) operators $H^\pm$ as above, the aim is to find necessary and sufficient conditions on the scattering data to correspond to some operator H in the class $\mathcal{B}_q(H^+, H^-)$.

In fact, the associated problem for the case $q = 1$ is well studied (see [12], [13], [15], [23], [24], [37]). However, as pointed out to us by A. K. Khanmamedov, the proof of Lemma 7.3 from [12] contains an error which consequently also affects our follow-up papers [13] and [15]. One aim of the present paper is to correct this error.

[1]Here, the signs "−" and "+" refer to the left and right half-axis, respectively.

Unfortunately, the properties of scattering data obtained below are not sufficient to solve the inverse scattering problem in $\mathcal{B}_1(H^+, H^-)$, but they are sufficient to reconstruct a unique operator H which belongs to the class $\mathcal{B}_0(H^+, H^-)$. For $q \geq 2$ this mismatch does not arise and we give necessary and sufficient conditions for the scattering data to correspond to an operator in the class $\mathcal{B}_q(H^+, H^-)$.

We emphasize that the mismatch for $q = 1$ only occurs if the limiting operators are genuine finite-gap operators and does not occur in the case of constant limiting operators. In particular, the approach used here allows us to solve the inverse scattering problem in the class $\mathcal{B}_1(H^+_{const}, H^-_{const})$ as a special case, when the background operators are constant and form a simple step,

$$(1.4) \qquad H^{\pm}_{const} f(n) = a^{\pm} f(n+1) + a^{\pm} f(n-1) + b^{\pm} f(n), \quad a^{\pm} > 0.$$

Moreover, in Section 4 we show that in the case $H^+_{const} = H^-_{const}$ the reflection coefficient is continuous at the edges of the spectrum in the resonance case for the finite first moment. Note, that for an arbitrary finite gap background $H^+ = H^-$ the continuity of the reflection coefficient at the edges of the spectrum for $q = 1$ remains an open question. The only case when this fact was established corresponds to a special type of the Jacobi operator (see [25]), which is associated with the Volterra (or Kac–van Moerbeke) lattice

$$(1.5) \qquad b(n) \equiv 0, \quad \sum |n|(|a(2n) - A_1| + |a(2n+1) - A_2|) < \infty, \quad A_1 \neq A_2.$$

Roughly, the scheme for solving the direct scattering problem for the case of a fixed finite moment q is the following:

Step 1. Construct transformation operators $K_{\pm}$ associated with both sides. Those operators intertwine between H and $H^{\pm}$ in the sense that $HK_{\pm} = K_{\pm}H^{\pm}$, respectively. Show that the corresponding kernels $K_{\pm}(n, m)$ are triangular.

Step 2. (a) Relate decay properties of the kernels of the transformation operators with q. (b) Derive formulas connecting these kernels with the coefficients $a(n), b(n)$. (c) Investigate the analytical properties of the Jost solutions with respect to q.

Step 3. Investigate the spectrum of H; derive the properties of the scattering matrix, in particular, its unitary property and asymptotical behavior at infinity; establish the connection between left and right scattering data.

Step 4. Derive the left and right Marchenko equations, which are the main equations of the inverse problem and connect the transformation operators to the set $\mathcal{S}(H)$ of scattering data (consisting of the scattering matrix, the discrete spectrum, and normalizing constants). The kernels $F_{\pm}$ of the Marchenko equations only depend on $\mathcal{S}(H)$.

Step 5. Establish decay properties of $F_{\pm}$ in terms of q.

The solution of the inverse scattering problem consists of the following steps:

Step 6. (a) Now we are given a set $\mathcal{S}$ of the same structure as the set of scattering data and with properties as described in Steps 3 and 5. This implies that we have given functions $F_{\pm}$ which satisfy the prescribed decay properties. These properties are sufficient to prove that the left and right Marchenko equations are uniquely solvable with respect to the kernels of the transformation operators. Invoking Step 2, (b), the kernels give rise to two sets of coefficients. (b) Both restored coefficients can in general be well controlled on the associated half-axis; on the opposite half-axis, they cannot be analyzed. To ensure that the solution lies within the class, one has to show that both sets of coefficients have the q'th moment finite on the associated side.

Step 7. This is technically the most challenging step. We prove a uniqueness theorem, that is, we show that the two restored sets of coefficients are equal and give rise to a unique Jacobi operator H with $\mathcal{S} = \mathcal{S}(H)$.

The pioneering work for this scattering problem was done by Kay and Moses [22] and mathematically rigorously by Faddeev [17]. They focussed on studying the main analytical properties of the scattering data (Step 3), on the derivation of the Marchenko equation (Step 4), and on proving its unique solvability (Step 6, (a)). Steps 1–2 and Step 6, (b), were already studied by Marchenko [29] when solving the scattering problem on the half-axis. However, later investigations by Deift and Trubowitz [10] showed that the properties of the scattering data listed in [17] are not sufficient to prove the uniqueness theorem (Step 7). An example for a set $\mathcal{S}$ was given there, which satisfied all conditions proposed in [17], but the restored potential did not have the required decaying behavior on one half-axis. It turned out that the behavior of the scattering data at the edge of the continuous spectrum plays a key role in the proof of the uniqueness theorem. For finite second moments of perturbation this behavior was described in [10]. There the characteristic properties of the scattering data were derived and the direct/inverse scattering problems were solved in the respective class of Schrödinger operators. But the approach used in [10] was not applicable for finite first moments. One has to emphasize, that a description of the characteristic properties of the scattering data for a class of potentials with finite first moments (these constitute the largest class of potentials for which direct/inverse scattering can be studied within the class) is a much more complicated problem. It was solved by Marchenko in 1977, [30]. The condition on the scattering data at the edge of the continuous spectrum is now referred to as Marchenko condition (see also the discussion before Theorem 3.3). Marchenko's approach for solving the inverse scattering problem became the classic method and was successfully generalized to several other types of operators and potentials, such as asymptotically periodic, finite-gap non-periodic, or steplike potentials.

In this note we study some features of Marchenko's method applied to Jacobi operators on finite-gap backgrounds. Scattering theory for Jacobi operators with constant background is a likewise classic topic with a long tradition. Originally developed on an informal level by Kac, Case, and Geronimo [6], [7], [8], the first rigorous results were given by Guseinov [19] with further extensions by Teschl [33], [34]. Moreover, these results have direct applications for solving the Toda lattice via the inverse scattering transform [34] and investigating its long-time asymptotics via the nonlinear steepest decent method (see, e.g., the review [27]). For the case of steplike backgrounds we refer to [2], [11] and for applications to the Toda lattice to [3], [4], [9], [36]. The case of (steplike) periodic backgrounds was considered in [12], [13], [15], [23], [24], [37] and applications to the Toda lattice can be found in [14], [16], [20], [21], [28], and [31].

2. Direct scattering problem

Steps 1–2. (Estimates on the transformation operators, properties of the Jost solutions, etc.). In the case of constant background operators, in particular, when they coincide with the discrete Laplacian $H^+_{const} = H^-_{const} = H_0$, where $H_0 y(n) = \frac{1}{2}\{y(n-1) + y(n+1)\}$, the transformation operators convert the "exponents" (the generalized eigenfunctions of H_0) into the Jost solutions. Namely, if $\lambda \in \mathbb{C}$ is a spectral parameter, then these eigenfunctions are $z^{\pm n}$, where $\lambda = \frac{1}{2}(z + z^{-1})$, $|z| \leq$

1. When the background operator is a finite-gap operator, the role of "exponents" is played by its Weyl solutions. Though the behavior of the finite-gap Weyl solutions remains exponential as $\lambda \to \infty$, they exhibit a much more complicated structure than the simple exponents.

We first recall some basic facts on the spectral analysis of finite-gap Jacobi operators[2]. Since we consider two different background operators, we introduce notations for both of them simultaneously. Let the spectra of $H^\pm$ in (1.2) consist of $r_\pm + 1$ bands,

$$\sigma_\pm := \sigma(H^\pm) = \bigcup_{j=0}^{r_\pm} [E_{2j}^\pm, E_{2j+1}^\pm], \quad E_0^\pm < E_1^\pm < \cdots < E_{2r_\pm+1}^\pm.$$

Associated with these sets are the Riemann surfaces of the functions

$$(2.1) \qquad P_\pm(\lambda) = - \prod_{j=0}^{2r_\pm+1} \sqrt{\lambda - E_j^\pm},$$

where $\sqrt{\cdot}$ denotes the standard branch of the square root cut along $(-\infty, 0)$. Each of the operators $H^\pm$ is uniquely defined by its spectrum $\sigma_\pm$ and the divisor of poles of its Weyl function (as a function on the respective Riemann surface)

$$(2.2) \qquad \sum_{j=1}^{r^\pm} (\mu_j^\pm, \sigma_j^\pm), \text{ where } \mu_j^\pm \in [E_{2j-1}^\pm, E_{2j}^\pm] \text{ and } \sigma_j^\pm \in \{-1, 1\}.$$

It is more convenient to consider the spectral parameter in the complex plane with cuts than working on two different Riemann surfaces. Therefore we identify $\mathbb{C} \setminus \sigma(H^+)$ with the upper sheet of the Riemann surface of P_+ and $\mathbb{C} \setminus \sigma(H^-)$ with the lower sheet of the Riemann surface of P_-. To distinguish the boundaries we denote the upper and lower points of the cuts along $\sigma_\pm$ by $\sigma_\pm^\mathrm{u}$ and $\sigma_\pm^\mathrm{l}$. Symmetric points of these cuts are denoted by λ^u and λ^l. In particular, for a function f we have

$$f(\lambda^\mathrm{u}) := \lim_{\varepsilon \downarrow 0} f(\lambda + \mathrm{i}\varepsilon), \quad f(\lambda^\mathrm{l}) := \lim_{\varepsilon \downarrow 0} f(\lambda - \mathrm{i}\varepsilon), \quad \lambda \in \sigma_\pm.$$

Let $\psi^\pm(\lambda, n)$ be the Weyl solutions of the spectral equations

$$(2.3) \qquad H^\pm \psi(\lambda, n) = \lambda \psi(\lambda, n), \quad \lambda \in \mathbb{C}, \quad n \in \mathbb{Z},$$

which are uniquely defined by the conditions $\psi^\pm(\lambda, 0) = 1$ and $\psi^\pm(\lambda, \cdot) \in \ell^2(\mathbb{Z}_\pm)$ as $\lambda \in \mathbb{C} \setminus \sigma_\pm$. Then the solution $\psi^+(\lambda)$ (resp. $\psi^-(\lambda)$) coincides with the upper (resp. lower) branch of the Baker-Akhiezer function of the operator H^+ (resp. H^-). The twin branches of these functions are denoted by $\breve{\psi}^\pm(\lambda)$. They are solutions of (2.3) as well, and satisfy $\breve{\psi}^\pm(\lambda, 0) = 1$ and $\breve{\psi}^\pm(\lambda, \cdot) \in \ell^2(\mathbb{Z}_\mp)$ as $\lambda \in \mathbb{C} \setminus \sigma_\pm$. The solutions $\psi^\pm(\lambda, n)$, $\breve{\psi}^\pm(\lambda, n)$ are continuous till the boundary for $\lambda \to \lambda^{\mathrm{u,l}} \in \sigma_\pm^{\mathrm{u,l}} \setminus \partial\sigma_\pm$ as functions of the spectral parameter λ, where we denote the set of spectral edges by $\partial\sigma_\pm := \{E_0^\pm, \ldots, E_{2r_\pm+1}^\pm\}$. Moreover, they satisfy the symmetry condition

$$(2.4) \qquad \psi^\pm(\lambda^\mathrm{l}, n) = \overline{\psi^\pm(\lambda^\mathrm{u}, n)} = \breve{\psi}^\pm(\lambda^\mathrm{u}, n), \quad \lambda \in \sigma_\pm, \quad n \in \mathbb{Z}.$$

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[2]For more details see Chapter 9 of [34].

To describe possible poles and other singularities of the Weyl solutions we distinguish the following disjoint subsets of $\{\mu_1^\pm, \ldots, \mu_{r_\pm}^\pm\}$ (compare (2.2))

$$(2.5) \qquad \begin{aligned} M^\pm &= \{\mu_j^\pm \,|\, \mu_j^\pm \in \mathbb{R} \setminus \sigma_\pm \text{ is a pole of } \psi^\pm(\lambda, 1)\}, \\ \hat{M}^\pm &= \{\mu_j^\pm \,|\, \mu_j^\pm \in \partial\sigma_\pm\}, \end{aligned}$$

and introduce auxiliary functions by

$$(2.6) \qquad \delta_\pm(\lambda) := \prod_{\mu_j^\pm \in M_\pm} (\lambda - \mu_j^\pm), \quad \hat{\delta}_\pm(\lambda) := \delta_\pm(\lambda) \prod_{\mu_j^\pm \in \hat{M}_\pm} \sqrt{\lambda - \mu_j^\pm},$$

where $\prod = 1$ if there are no multipliers. Accordingly, we set

$$(2.7) \qquad \tilde{\psi}^\pm(\lambda, n) = \delta_\pm(\lambda)\psi^\pm(\lambda, n), \quad \hat{\psi}^\pm(\lambda, n) = \hat{\delta}_\pm(\lambda)\psi^\pm(\lambda, n).$$

Then $\tilde{\psi}^\pm(\lambda, n)$ have no poles in open gaps of the spectra, satisfy the symmetry property (2.4), and have square root singularities at the points of the sets $\hat{M}^\pm$, whereas $\hat{\psi}^\pm(\lambda, n)$ are continuous on $\mathbb{C} \setminus \sigma_\pm$ till the boundaries, but in general violate the symmetry property.

Introduce the discrete Wronskians of the Weyl solutions by

$$(2.8) \qquad W^\pm(\lambda) := a^\pm(n)\big(\check{\psi}^\pm(\lambda, n)\psi^\pm(\lambda, n+1) - \check{\psi}^\pm(\lambda, n+1)\psi^\pm(\lambda, n)\big).$$

Then

$$(2.9) \qquad W^\pm(\lambda) = \pm\frac{1}{g_\pm(\lambda)}, \quad \text{where} \quad g_\pm(\lambda) := \frac{\prod_{j=1}^{r_\pm}(\lambda - \mu_j^\pm)}{P_\pm(\lambda)}$$

are the Green functions of the operators $H^\pm$ on the main diagonal at point $(0,0)$. By the choice of the square root branch in $P_\pm(\lambda)$ the Green functions satisfy

$$(2.10) \qquad \text{Im}(g_\pm(\lambda^{\mathrm{u}})) > 0, \quad \text{Im}(g_\pm(\lambda^{\mathrm{l}})) < 0, \quad \lambda \in \sigma_\pm.$$

The functions $\psi^\pm(\lambda, \cdot)$ form complete orthogonal systems on the spectra $\sigma_\pm$ with respects to the weights

$$(2.11) \qquad d\omega_\pm(\lambda) = \frac{1}{2\pi\mathrm{i}}g_\pm(\lambda)d\lambda.$$

Namely,

$$(2.12) \qquad \oint_{\sigma_\pm} \overline{\psi^\pm(\lambda, m)}\psi^\pm(\lambda, n)d\omega_\pm(\lambda) = \delta(n, m),$$

where $\delta(n, m)$ is the Kronecker symbol and

$$(2.13) \qquad \oint_{\sigma_\pm} f(\lambda)d\lambda := \int_{\sigma_\pm^{\mathrm{u}}} f(\lambda^{\mathrm{u}})d\lambda - \int_{\sigma_\pm^{\mathrm{l}}} f(\lambda^{\mathrm{l}})d\lambda.$$

Next, we describe the properties of the transformation operators on finite-gap backgrounds, following [23] and [12] with a small correction in the steplike case. Assume that the perturbation coefficients $\{a(n)-a^\pm(n), b(n)-b^\pm(n)\}$ have minimal (i.e. first) finite moments on the half-axes. Evidently, all remains valid for finite higher moments.

Lemma 2.1. *Let $H \in \mathcal{B}_1(H^+, H^-)$. Then there exist Jost solutions $\phi_\pm(\lambda, n)$ of the spectral problem*

$$(2.14) \qquad H\phi(\lambda, n) = \lambda\phi(\lambda, n)$$

which are asymptotically close to the Weyl solutions $\psi^\pm(\lambda, n)$ of the background operators $H^\pm$ as $n \to \pm\infty$. These solutions can be represented as

$$(2.15) \qquad \phi_\pm(\lambda, n) = \sum_{m=n}^{\pm\infty} K_\pm(n, m)\psi^\pm(\lambda, m), \quad \lambda \in \mathbb{C},$$

where the kernels $K_\pm(n, \cdot)$ of the transformation operators are real valued and satisfy for $\pm m > \pm n$

$$(2.16) \quad |K_\pm(n, m)| \le C_\pm(n) \prod_{j=n-{}^0_1}^{\pm\infty} \frac{a(j)}{a^\pm(j)} \sum_{j=[\frac{m+n}{2}]}^{\pm\infty} \Big(|a(j) - a^\pm(j)| + |b(j) - b^\pm(j)|\Big).$$

The functions $C_\pm(n) > 1$ decay as $n \to \pm\infty$.

Note that in the steplike case, $C_\pm(n)$ grow with the order of the products in (2.16) as $n \to \mp\infty$. In the case $H^+ = H^-$ they remain bounded and are usually replaced by constants. From (2.15) and (2.12) it follows that

$$K_\pm(n, m) = \oint_{\sigma_\pm} \phi_\pm(\lambda, n)\breve{\psi}_\pm(\lambda, m)d\omega_\pm(\lambda), \quad \pm m \ge \pm n,$$

and $K_\pm(n, m) = 0$ for $\pm m < \pm n$. Therefore,

$$(2.17) \quad \begin{aligned} &a(n-1)K_\pm(n-1, m) + b(n)K_\pm(n, m) + a(n)K_\pm(n+1, m) \\ &= a^\pm(m-1)K_\pm(n, m-1) + b^\pm(m)K_\pm(n, m) + a^\pm(m)K_\pm(n, m+1). \end{aligned}$$

Evaluating (2.17) at $m = n \mp 1$ and $m = n$ we obtain

$$(2.18) \qquad a(n) = a^+(n)\frac{K_+(n+1, n+1)}{K_+(n, n)} = a^-(n)\frac{K_-(n, n)}{K_-(n+1, n+1)},$$

$$(2.19) \quad \begin{aligned} b(n) &= b^+(n) + a^+(n)\frac{K_+(n, n+1)}{K_+(n, n)} - a^+(n-1)\frac{K_+(n-1, n)}{K_+(n-1, n-1)}, \\ b(n) &= b^-(n) + a^-(n-1)\frac{K_-(n, n-1)}{K_-(n, n)} - a^-(n)\frac{K_-(n+1, n)}{K_-(n+1, n+1)}. \end{aligned}$$

With these formulas we can compute the coefficients of the perturbed operator H via the kernels of the left and right transformation operators.

For $\lambda \in \sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}}$, equations (2.14) also have solutions

$$\breve{\phi}_\pm(\lambda, n) = \sum_{m=n}^{\pm\infty} K_\pm(n, m)\breve{\psi}^\pm(\lambda, m), \quad \lambda \in \sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}},$$

which cannot be continued to the complex plane under condition (1.3) for fixed q. For $\lambda \in \sigma_\pm$, we have $\breve{\phi}_\pm(\lambda, n) = \overline{\phi_\pm(\lambda, n)}$, and (1.3), (2.15), and (2.8) imply

$$(2.20) \qquad W(\overline{\phi_\pm(\lambda)}, \phi_\pm(\lambda)) = W^\pm(\lambda) = \pm g_\pm(\lambda)^{-1}.$$

For any $q = 1, 2, \ldots$, due to (2.15) the Jost solutions $\phi_\pm$ inherit almost all properties of their background Weyl solutions, in particular, the structure of poles and square root singularities. But at the edges of the spectra $\partial\sigma_\pm$ the influence of the moment q is already perceptible. For $q = 2, 3, \ldots$, the Jost solutions are differentiable functions of the local parameter $\sqrt{\lambda - E}$ at $E \in \partial\sigma_\pm$, but for $q = 1$ they cease to be. This complicates investigating the scattering data at the edges of the spectra.

We turn to *Step 3* next. Define in analogy to (2.6), (2.7)

$$(2.21) \qquad \tilde{\phi}_\pm(\lambda, n) = \delta_\pm(\lambda)\phi_\pm(\lambda, n), \quad \hat{\phi}_\pm(\lambda, n) = \hat{\delta}_\pm(\lambda)\phi_\pm(\lambda, n),$$

and introduce the sets

$$(2.22) \qquad \sigma := \sigma_+ \cup \sigma_-, \quad \sigma^{(2)} := \sigma_+ \cap \sigma_-, \quad \sigma_\pm^{(1)} = \text{clos}\left(\sigma_\pm \setminus \sigma^{(2)}\right).$$

Then σ is the absolutely continuous spectrum of H and $\sigma_+^{(1)} \cup \sigma_-^{(1)}$ and $\sigma^{(2)}$ are the spectra of multiplicity one and two (cf. [15]). Denote by $\text{int}(\sigma)$ the sets of inner points of the spectrum, $\text{int}(\sigma) := \sigma \setminus \partial\sigma$. In addition to the continuous spectrum, H has a finite number of eigenvalues $\sigma_d = \{\lambda_1, \ldots, \lambda_p\} \subset \mathbb{R} \setminus \sigma$ (finiteness of σ_d is proven in [32]). We introduce normalizing constants by

$$(2.23) \qquad \gamma_{\pm,k}^{-1} = \sum_{n \in \mathbb{Z}} |\tilde{\phi}_\pm(\lambda_k, n)|^2, \quad 1 \le k \le p.$$

The Wronskian of two Jost solutions, given by

$$(2.24) \qquad W(\lambda) := a(n)\left(\phi_-(n)\phi_+(n+1) - \phi_-(n+1)\phi_+(n)\right),$$

is meromorphic on $\mathbb{C} \setminus \sigma$, since the Jost solutions are meromorphic there, with possible poles on $M_+ \cup M_- \cup (\hat{M}_+ \cap \hat{M}_-)$ and possible square root singularities on $\hat{M}_+ \cup \hat{M}_- \setminus (\hat{M}_+ \cap \hat{M}_-)$. Consider the scattering relations

$$(2.25) \qquad T_\mp(\lambda)\phi_\pm(\lambda, n) = \overline{\phi_\mp(\lambda, n)} + R_\mp(\lambda)\phi_\mp(\lambda, n), \quad \lambda \in \sigma_\mp^{\text{u,l}}.$$

For steplike cases, the entries of the scattering matrix generally live on different sets. The left reflection coefficient R_- and transmission coefficient T_- are defined on the upper and lower sides of σ_-, the right coefficients R_+ and T_+ are defined on the sides of σ_+. The scattering data $\mathcal{S}$ has the following structure,

$$(2.26) \qquad \begin{aligned} \mathcal{S} = \big\{ &R_+(\lambda), T_+(\lambda), \lambda \in \sigma_+^{\text{u,l}}; R_-(\lambda), T_-(\lambda), \lambda \in \sigma_-^{\text{u,l}}; \\ &\lambda_1, \ldots, \lambda_p \in \mathbb{R} \setminus (\sigma_+ \cup \sigma_-), \gamma_{\pm,1}, \ldots, \gamma_{\pm,p} \in \mathbb{R}_+ \big\}. \end{aligned}$$

The entries of $\mathcal{S}$ are not independent of each other, their dependencies are collected in the next lemma (see [15]). These results hold for any fixed $q = 1, 2, \ldots$ and might be specified for $q = 2, 3, \ldots$, which is not of interest here. Since the conditions in this list are not independent, some of them are partly covered by others. The dependence of $\mathcal{S}$ on q will be expressed by property $\mathbf{IV_q}$ in Lemma 2.4 below.

Lemma 2.2. *Let $H \in \mathcal{B}_q(H^+, H^-)$ for fixed $q = 1, 2, \ldots$. Then the entries of the scattering matrix of H have the following properties*
  I.

$$\begin{aligned} &\text{(a)} \quad T_\pm(\lambda^{\text{u}}) = \overline{T_\pm(\lambda^{\text{l}})}, \; R_\pm(\lambda^{\text{u}}) = \overline{R_\pm(\lambda^{\text{l}})} && \lambda \in \sigma_\pm, \\ &\text{(b)} \quad T_\pm(\lambda)\overline{T_\pm^{-1}(\lambda)} = R_\pm(\lambda), && \lambda \in \sigma_\pm^{(1)}, \\ &\text{(c)} \quad 1 - |R_\pm(\lambda)|^2 = g_\pm(\lambda)g_\mp^{-1}(\lambda)|T_\pm(\lambda)|^2, && \lambda \in \sigma^{(2)}, \\ &\text{(d)} \quad \overline{R_\pm(\lambda)}T_\pm(\lambda) + R_\mp(\lambda)\overline{T_\pm(\lambda)} = 0, && \lambda \in \sigma^{(2)}. \end{aligned}$$

  II. *The functions $T_\pm(\lambda)$ can be continued analytically in $\mathbb{C} \setminus (\sigma \cup M_\pm \cup \check{M}_\pm \cup \sigma_d)$, where they satisfy*

$$(2.27) \qquad \frac{1}{T_+(\lambda)g_+(\lambda)} = \frac{1}{T_-(\lambda)g_-(\lambda)} =: W(\lambda).$$

The function $W(\lambda)$ has the following properties:

(a) $\tilde{W}(\lambda) = \delta_+(\lambda)\delta_-(\lambda)W(\lambda)$ *is holomorphic in* $\mathbb{C} \setminus \sigma$ *with simple zeros at* λ_k,

$$(2.28) \qquad \left(\frac{d\tilde{W}}{d\lambda}(\lambda_k)\right)^2 = \gamma_{+,k}^{-1}\gamma_{-,k}^{-1}.$$

In addition, $\overline{\tilde{W}(\lambda^{\mathrm{u}})} = \tilde{W}(\lambda^{\mathrm{l}})$ *as* $\lambda \in \sigma$ *and* $\tilde{W}(\lambda) \in \mathbb{R}$ *as* $\lambda \in \mathbb{R} \setminus \sigma$.

(b) $\hat{W}(\lambda) = \hat{\delta}_+(\lambda)\hat{\delta}_-(\lambda)W(\lambda)$ *is continuous on* $\mathbb{C} \setminus \sigma$ *up to the boundary* $\sigma^{\mathrm{u}} \cup \sigma^{\mathrm{l}}$. *It can have zeros on* $\Sigma_v := \partial\sigma \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)})$[3] *and does not vanish at any other point of* σ. *If* $\hat{W}(E) = 0$ *for* $E \in \Sigma_v$, *then*

$$(2.29) \qquad \hat{W}^{-1}(\lambda) = O\big((\lambda - E)^{-1/2}\big), \quad \textit{for } \lambda \in \sigma \textit{ close to } E,$$

$$(2.30) \qquad \hat{W}^{-1}(\lambda) = O\big((\lambda - E)^{-1/2-\varepsilon}\big), \quad \textit{for } \lambda \in \mathbb{C} \setminus \sigma \textit{ close to } E.$$

(c) $T_+(\infty) = T_-(\infty) > 0$.

III.

(a) *The reflection coefficients* $R_\pm(\lambda)$ *are continuous functions on* $\mathrm{int}(\sigma_\pm^{\mathrm{u,l}})$.
  If $E \in \partial\sigma_+ \cup \partial\sigma_-$ *and* $\hat{W}(E) \neq 0$, *then* $R_\pm(\lambda)$ *are also continuous at* E.

(b) *If* $E \in \partial\sigma_+ \cap \partial\sigma_-$ *and* $\hat{W}(E) \neq 0$, *then*

$$(2.31) \qquad R_\pm(E) = \begin{cases} -1 & \textit{as} \quad E \notin \hat{M}_\pm, \\ 1 & \textit{as} \quad E \in \hat{M}_\pm. \end{cases}$$

Remarks.

- For any fixed $q = 2, 3, \ldots$, properties (2.29) and (2.30) can be replaced by (see [5]): *If* $\hat{W}(E) = 0$ *for* $E \in \Sigma_v$, *then* $\hat{W}(\lambda) = C\sqrt{\lambda - E}(1 + o(1))$, $C \neq 0$, *for* $\lambda \to E$.
- For any fixed $q = 2, 3, \ldots$, property **III**, **(a)** is given by: *The reflection coefficients* $R_\pm(\lambda)$ *are continuous functions on* $\sigma_\pm^{\mathrm{u,l}}$.

Steps 4–5 (The Marchenko equations and estimates on their kernels)

Theorem 2.3. ([15]) *Let* $H \in \mathcal{B}_q(H^+, H^-)$ *for a fixed* $q = 1, 2, \ldots$. *Then the kernels of the transformation operators satisfy the Marchenko equations*

$$(2.32) \qquad K_\pm(n, m) + \sum_{\ell=n}^{\pm\infty} K_\pm(n, \ell)F_\pm(\ell, m) = \frac{\delta(n, m)}{K_\pm(n, n)}, \quad \pm m \geq \pm n,$$

where

$$(2.33) \qquad \begin{aligned} F_\pm(m, n) &= \oint_{\sigma_\pm} R_\pm(\lambda)\psi^\pm(\lambda, m)\psi^\pm(\lambda, n)d\omega_\pm \\ &+ \int_{\sigma_\mp^{(1),\mathrm{u}}} |T_\mp(\lambda)|^2\psi^\pm(\lambda, m)\psi^\pm(\lambda, n)d\omega_\mp + \sum_{k=1}^p \gamma_{\pm,k}\tilde{\psi}^\pm(\lambda_k, n)\tilde{\psi}^\pm(\lambda_k, m). \end{aligned}$$

We abbreviate

$$(2.34) \qquad \kappa_\pm(n, m) = \frac{K_\pm(n, m)}{K_\pm(n, n)}, \quad \pm m > \pm n,$$

---

[3]This is the set of possible virtual levels of H.

and rewrite the Marchenko equations (2.32) by

$$(2.35) \qquad \kappa_\pm(n,m) + F_\pm(n,m) + \sum_{\ell=n\pm1}^{\pm\infty} \kappa_\pm(n,\ell)F_\pm(\ell,m) = 0, \quad \pm m > \pm n,$$

$$(2.36) \qquad 1 + F_\pm(n,n) + \sum_{\ell=n\pm1}^{\pm\infty} \kappa_\pm(n,\ell)F_\pm(\ell,n) = \frac{1}{K_\pm(n,n)^2}.$$

The last remaining property to complete the list of necessary and sufficient conditions on the scattering data depends on q.

Lemma 2.4. *Let $H \in \mathcal{B}_q(H^+, H^-)$ for a fixed $q = 1, 2, \ldots$. Then*

$\mathbf{IV_q}$*. The functions $F_\pm(n,m)$ defined by (2.33) satisfy the following conditions.*

(i) *There exist functions $p_\pm(n) \geq 0$, $C_\pm(n) > 0$ such that $|n|^q p_\pm(n) \in \ell^1(\mathbb{Z}_\pm)$, $C_\pm(n)$ do not increase as $n \to \pm\infty$, and such that*

$$(2.37) \qquad |F_\pm(n,m)| \leq C_\pm(n) \sum_{j=n+m}^{\pm\infty} p_\pm(j).$$

(ii) *The following estimates hold*

$$(2.38) \qquad \sum_{n=n_0}^{\pm\infty} |n|^\alpha \big| F_\pm(n,n) - F_\pm(n\pm1, n\pm1) \big| < \infty,$$

$$(2.39) \qquad \sum_{n=n_0}^{\pm\infty} |n|^\alpha \big| a^\pm(n) F_\pm(n, n+1) - a^\pm(n-1) F_\pm(n-1, n) \big| < \infty,$$

where $\alpha = q$ as $q = 2, 3, \ldots$ and $\alpha = 0$ as $q = 1$.

Proof. **(i)** The considerations for the left and right half axes are analogous, so we prove (2.37) for the "+" case and drop the index. For $m > n$, we set in (2.35) $\kappa := \kappa_+$, $F := F_+$, and let $\hat{a} := a^+$, $\hat{b} := b^+$, and $C(n) = C_+(n)$, where $C_+(n) \geq 1$ is the function from estimate (2.16). Then

$$(2.40) \qquad\qquad\qquad |\kappa(n,m)| < \sigma(n+m),$$

where

$$(2.41) \qquad \sigma(n) := C(n) \sum_{\ell=\left[\frac{n}{2}\right]}^{\infty} \big(|a(\ell) - \hat{a}(\ell)| + |b(\ell) - \hat{b}(\ell)| \big).$$

Define $\sigma_1(n) := \sum_{\ell=n}^{\infty} \sigma(\ell)$, then

$$n^{q-1}\sigma(n) \in \ell^1(\mathbb{Z}_+), \quad q = 1, 2, \ldots, \quad n^{q-2}\sigma_1(n) \in \ell^1(\mathbb{Z}_+), \quad q = 2, 3, \ldots.$$

Moreover,

$$(2.42) \qquad\qquad \sigma_1(n) \geq \sigma_1(\ell), \quad \sigma(n) \geq \sigma(\ell), \quad n < \ell,$$

and since

$$\frac{\sigma_1^s(n) - \sigma_1^s(n+1)}{s!} = (\sigma_1(n) - \sigma_1(n+1)) \frac{\sigma_1^{s-1}(n) + \cdots + \sigma_1^{s-1}(n+1)}{s!},$$

then

$$(2.43) \qquad \frac{\sigma_1^s(n) - \sigma_1^s(n+1)}{s!} \geq (\sigma_1(n) - \sigma_1(n+1)) \frac{\sigma_1^{s-1}(n+1)}{(s-1)!}.$$

To obtain the required estimate, we apply the method of successive approximations to (2.35). Let

$$F^{(0)}(n,m) = -\kappa(n,m), \quad F^{(s)}(n,m) = -\sum_{\ell=n+1}^{\infty} \kappa(n,\ell)F^{(s-1)}(\ell,m).$$

We prove by induction that

$$(2.44) \qquad |F^{(s)}(n,m)| \leq \sigma(n+m)\frac{\sigma_1^s(2n+1)}{s!}.$$

By (2.40) and (2.41), this estimate is true for $s = 0$. For $s \geq 1$, take into account (2.42) and (2.43), then

$$\begin{aligned}
|F^{(s)}(n,m)| &\leq \sum_{\ell=n+1}^{\infty} \sigma(n+\ell)\sigma(\ell+m)\frac{\sigma_1^{s-1}(2\ell+1)}{(s-1)!} \\
&\leq \sigma(n+m)\sum_{\ell=n+1}^{\infty} \sigma(n+\ell)\frac{\sigma_1^{s-1}(n+\ell+1)}{(s-1)!} \\
&= \sigma(n+m)\sum_{\ell=n+1}^{\infty} (\sigma_1(n+\ell)-\sigma_1(n+\ell+1))\frac{\sigma_1^{s-1}(n+\ell+1)}{(s-1)!} \\
&\leq \sigma(n+m)\sum_{\ell=n+1}^{\infty} \frac{\sigma_1^s(n+\ell)-\sigma_1^s(n+\ell+1)}{s!} = \sigma(n+m)\frac{\sigma_1^s(2n+1)}{s!}.
\end{aligned}$$

Inequality (2.44) implies

$$(2.45) \qquad |F(n,m)| \leq \sigma(n+m)\exp\{\sigma_1(2n+1)\} \leq C_1(n)\sigma(n+m), \quad m > n,$$

where $C_1(n)$ is of the same type as $C_+(n)$ in (2.16).

For $n = m$, we use the symmetry of the kernel $F(n,m)$ and (2.18), (2.40), (2.44), and (2.36). Then

$$\begin{aligned}
|F(n,n)| &\leq \left|\prod_{\ell=n}^{\infty} \frac{a(\ell)^2}{\hat{a}(\ell)^2} - 1\right| + \sum_{\ell=n+1}^{\infty} \sigma^2(n+\ell) \\
&\leq C(e^{C\sigma(2n)} - 1) + \sigma(2n+1)\sigma_1(2n+1),
\end{aligned}$$

that is,

$$(2.46) \qquad |F(n,n)| \leq C_2(n)\sigma(2n),$$

where $C_2(n)$ again is a function similar to $C_+(n)$ in (2.16), since $\sigma_1(n)$ increases for $n \to -\infty$. This yields (2.37) if we define $p_+(\ell) = |a(\ell) - \hat{a}(\ell)| + |b(\ell) - \hat{b}(\ell)|$ and $C_+(n) = \max\{C_1(n), C_2(n)\}$.

(ii) Let us first show that (2.39) are satisfied for $q = 2, 3, \dots$. By (2.18) we have

$$(2.47) \qquad b(n) - b^+(n) = a^+(n)\kappa_+(n,n+1) - a^+(n-1)\kappa_+(n-1,n),$$

$$(2.48) \qquad b(n) - b^-(n) = a^-(n-1)\kappa_-(n,n-1) - a^-(n)\kappa_-(n+1,n).$$

Consider the "+" case of (2.35), multiply it at $(n,n+1)$ by $n^q a^+(n)$ and at $(n-1,n)$ by $n^q a^+(n-1)$, and subtract these two equations from each other. Then using

(2.47), (2.40), (2.45), and monotonicity of $\sigma(n)$ we obtain

$$\sum_{n=n_0}^{+\infty} |n|^q \big| a^+(n) F_+(n, n+1) - a^+(n-1) F_+(n-1, n) \big|$$

$$\leq \sum_{n=n_0}^{+\infty} |n|^q \big| b(n) - b^+(n) \big| + 2C_1(n_0) \Big(\max_{j \in \mathbb{Z}} a^+(j) \Big) \sum_{n=n_0}^{+\infty} \sum_{\ell=n}^{+\infty} n^q \sigma(n-1+\ell) \sigma(n+\ell)$$

$$\leq \tilde{C}(n_0) \Big(1 + \sum_{n=n_0}^{+\infty} n\sigma(2n) \sum_{\ell=n_0}^{+\infty} (n+\ell)^{q-1} \sigma(n+\ell) \Big) < \infty.$$

(2.49)

Note that for $q = 1$ the last inequality in (2.49) does not hold, since $n\sigma(n) \notin \ell^1(\mathbb{Z}_+)$. It is for this reason that we multiply by n^0 in (**ii**). The "$-$" case of (2.39) is obtained analogously for $q = 2, 3, \dots$, using (2.48), (2.40), (2.45), and the symmetry $F_-(n, m) = F_-(m, n)$. To prove (2.38), we use (2.18) and (1.3) to obtain

$$\left| \frac{1}{K_+(n,n)^2} - \frac{1}{K_+(n+1,n+1)^2} \right| \leq \frac{1}{a_+(n)^2} \prod_{j=n+1}^{+\infty} \frac{a(j)^2}{a^+(j)^2} |a^+(n)^2 - a(n)^2|$$

$$\leq |a(n) - a^+(n)| \, e^{2\sigma_1(n)} \sup_{j \in \mathbb{Z}} \frac{|a(j) + a^+(j)|}{a^+(j)^2}.$$

Using this estimate and (2.36) we get (2.38) by the same reasoning as in (2.49). $\quad\square$

Theorem 2.5. *Let $H \in \mathcal{B}_q(H^+, H^-)$ for a fixed $q = 1, 2, \dots$. Then the scattering data $\mathcal{S}$ of H given by (2.26) satisfy properties* **I**–**III** *of Lemma 2.2. The functions $F_\pm(n, m)$ defined by (2.33) satisfy condition* **IV$_\mathbf{q}$** *of Lemma 2.4.*

It turns out, that conditions **I**–**IV$_\mathbf{q}$** are necessary and sufficient for solving the direct/inverse problem in the class $\mathcal{B}_q(H^+, H^-)$ with $q = 2, 3, \dots$. For $q = 1$, they are sufficient to solve the inverse scattering problem in $\mathcal{B}_0(H^+, H^-)$. In the next section we will prove the sufficiency of these conditions and describe the solution algorithm.

3. Inverse scattering problem

Step 6. Let $H^\pm$ be two finite-gap Jacobi operators associated with $a^\pm(n)$, $b^\pm(n)$. Correspondingly, σ_+, σ_-, $\psi^\pm$, and $\tilde{\psi}^\pm$ are given. Let $\mathcal{S}$ be a set of the form (2.26), which satisfies conditions **I**–**III**, and introduce $F_\pm(n, m)$ by (2.33). It follows from **I**–**III** that $F_\pm(n, m)$ are well defined. We now assume that $F_\pm(n, m)$ satisfy property **IV$_\mathbf{q}$**, (**i**) for a fixed $q = 1, 2, \dots$ and show that the Marchenko equations (2.32) are uniquely solvable with respect to $K_\pm(n, m)$.

Lemma 3.1. *Under condition* **IV$_\mathbf{q}$**, (**i**) *for a fixed $q = 1, 2, \dots$, equations (2.35) have unique solutions $\kappa_\pm(n, \cdot) \in \ell^1(n \pm 1, \pm\infty)$ satisfying*

$$(3.1) \qquad |\kappa_\pm(n, m)| \leq \hat{C}_\pm(n) \sum_{j=n+m}^{\pm\infty} p_\pm(j), \quad \pm m > \pm n.$$

Here $p_\pm(n) \geq 0$ are the same functions as in **IV$_\mathbf{q}$**, (**i**)*, and $\hat{C}_\pm(n) > 0$ are bounded as $n \to \pm\infty$.*

Proof. Equations (2.35) are generated by compact operators, hence for unique solvability it is sufficient to prove that the homogeneous equations

$$(3.2) \qquad f(m) + \sum_{\ell=n}^{\pm\infty} F_\pm(\ell, m) f(\ell) = 0$$

only have trivial solutions in $\ell^1(n, \pm\infty)$. The proof is the same for both cases, we will consider the "+" case. Suppose that $f(\ell)$, $\ell > n$, is a nontrivial solution of (3.2) and set $f(\ell) = 0$ for $\ell \le n$. Since $F_+(\ell, n) \in \mathbb{R}$, we can assume that $f(\ell) \in \mathbb{R}$. Denote by

$$(3.3) \qquad \widehat{f}(\lambda) = \sum_{m \in \mathbb{Z}} \psi^+(\lambda, m) f(m)$$

the generalized Fourier transform which is generated by the spectral decomposition (2.12) (cf. [35]). Recall that $\widehat{f}(\lambda) \in L^1_{loc}(\sigma^{\mathrm{u}}_+ \cup \sigma^{\mathrm{l}}_+)$. Multiplying (3.2) by $f(m)$, summing over $m \in \mathbb{Z}$, and using consecutively (2.12), (2.33), (3.3), and **I, (a)**, we arrive at

$$(3.4) \qquad \begin{aligned} & 2\int_{\sigma^u_+} |\widehat{f}(\lambda)|^2 d\omega_+(\lambda) + 2\operatorname{Re}\int_{\sigma^u_+} R_+(\lambda)\widehat{f}(\lambda)^2 d\omega_+(\lambda) \\ & + \int_{\sigma^{(1),u}_-} \widehat{f}(\lambda)^2 |T_-(\lambda)|^2 d\omega_-(\lambda) + \sum_{k=1}^p \gamma_{+,k} \left(\sum_{n \in \mathbb{Z}} \tilde{\psi}^+(\lambda_k, n) f(n) \right)^2 = 0. \end{aligned}$$

Since $\widehat{f}(\lambda) \in \mathbb{R}$ as $\lambda \in \sigma^{(1)}_-$ and $\tilde{\psi}^+(\lambda_k) \in \mathbb{R}$, the last two summands in (3.4) are nonnegative. The integrands of the first two summands are estimated by

$$|\widehat{f}(\lambda)|^2 + \operatorname{Re} R_+(\lambda)\widehat{f}(\lambda)^2 \ge |\widehat{f}(\lambda)|^2 - |R_+(\lambda)\widehat{f}(\lambda)^2| \ge \left(1 - |R_+(\lambda)|\right)|\widehat{f}(\lambda)|^2.$$

Dropping the last summand in (3.4) and using

$$\int_{\sigma^{(1),\mathrm{u}}_+} (1 - |R_+(\lambda)|)|\widehat{f}(\lambda)|^2 d\omega_+(\lambda) = 0,$$

which follows from **I, (b)**, yields

$$(3.5) \qquad 2\int_{\sigma^{(2),\mathrm{u}}} (1 - |R_+(\lambda)|)|\widehat{f}(\lambda)|^2 d\omega_+(\lambda) + \int_{\sigma^{(1),\mathrm{u}}_-} \widehat{f}(\lambda)^2 |T_-(\lambda)|^2 d\omega_-(\lambda) \le 0.$$

Since $|R_+(\lambda)| < 1$ and $\omega'_+(\lambda) > 0$ for $\lambda \in \operatorname{int}(\sigma^{(2)})$ and since $\omega'_-(\lambda) > 0$ for $\lambda \in \operatorname{int}(\sigma^{(1)}_-)$, we have

$$(3.6) \qquad \widehat{f}(\lambda) = 0, \quad \lambda \in \sigma^{(2)} \cup \sigma^{(1)}_- = \sigma_-.$$

The function $\widehat{f}(\lambda)$ defined by (3.3) is meromorphic on $\mathbb{C} \setminus \sigma_+$. But (3.6) implies that $\widehat{f}(\lambda)$ is in fact meromorphic on $\mathbb{C} \setminus \sigma^{(1)}_+$ and vanishes on σ_-. Hence $\widehat{f}(\lambda) = 0$, therefore $f(m) = 0$ as desired, and (2.35) has a unique solution in $\ell^1(n+1, +\infty)$.

To obtain estimate (3.1), we first introduce the following operator in $\ell^1(n, +\infty)$,

$$\left((I + \mathcal{F}^+_n)f\right)(m) := f(m) + \sum_{\ell=n}^{+\infty} F_+(\ell, m) f(\ell).$$

We already proved that the left-hand side of (3.4), which is equal to the scalar product $((I + \mathcal{F}^+_n)f, f)$, is nonnegative. Indeed, it is positive. If the spectrum of multiplicity two is nonempty, that is, if there exists a set where $|R_+| < 1$, then the

sum of the first two summands in (3.4) is positive and the other two summands are nonnegative. If $\sigma^{(2)} = \emptyset$, then necessarily the summand which corresponds to integrating over the set $\sigma_-^{(1),u}$ is positive. Hence, the operator $(I + \mathcal{F}_n^+)$ is compact and positive, and has a bounded inverse which we abbreviate by

$$\|(I + \mathcal{F}_n^+)^{-1}\| = \hat{C}^+(n) > 0.$$

Note that $\hat{C}^+(n)$ is not monotonic, there exists

$$(3.7) \qquad \max_{n > n_0} \hat{C}^+(n) = C_0^+.$$

Let $C_+(n)$ and $p_+(n)$ be the functions of (the "+" case of) estimate (2.37) and set

$$(3.8) \qquad \sigma^+(n) := C_+(n) \sum_{\ell=n}^{+\infty} p^+(\ell), \quad \sigma_1^+(n) := \sum_{\ell=n}^{+\infty} \sigma^+(\ell).$$

It follows from the Marchenko equation (2.35) that

$$\kappa_+(n, n+j) = -(I + \mathcal{F}_{n+1}^+)^{-1} F_+(n, n+j), \quad j = 1, 2, \ldots,$$

that is,

$$\sum_{j=1}^{+\infty} |\kappa(n, n+j)| \le \hat{C}^+(n) \sum_{j=1}^{+\infty} \sigma^+(2n+j) = \hat{C}^+(n) \sigma_1^+(2n+1).$$

Applying this inequality to (2.35) and using (2.37), (3.8), we obtain

$$|\kappa_+(n,m)| \le |F_+(n,m)| + \sum_{j=1}^{+\infty} |\kappa_+(n, n+j)| \sigma^+(n+j+m)$$

$$\le \sigma^+(n+m)(1 + \hat{C}^+(n) \sigma_1^+(2n+1)) \le \hat{C}_+(n) \sigma^+(n+m),$$

which proves (3.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Summarizing, we constructed $\kappa_\pm(n,m)$ for any n and $\pm m > \pm n$. To complete the reconstruction of the operators $K_\pm(n,m)$, $\pm m \ge \pm n$, it is necessary to verify that the left hand side of (2.36) is a positive function. It allows us to define $K_\pm(n,n)$ as a positive value, details of the proof can be found in [12].

Now we denote the following four sequences by $a_\pm$, $b_\pm$, $n \in \mathbb{Z}$,

$$(3.9) \qquad \begin{aligned} a_+(n) &= \frac{a^+(n) K_+(n+1, n+1)}{K_+(n,n)}, \quad a_-(n) = a^-(n) \frac{a^-(n) K_-(n,n)}{K_-(n+1, n+1)}, \\ b_+(n) &= b^+(n) + a^+(n) \kappa_+(n, n+1) - a^+(n-1) \kappa_+(n-1, n), \\ b_-(n) &= b^-(n) + a^-(n-1) \kappa_-(n, n-1) - a^-(n) \kappa_-(n+1, n), \end{aligned}$$

and investigate their asymptotical behavior for large $|n|$.

Lemma 3.2. *Let $F_{\pm}(n, m)$ satisfy condition $\mathbf{IV_q}$ for a fixed $q = 1, 2, \ldots$. Then the solutions of the Marchenko equation (2.32) satisfy*

$$\sum_{n=n_0}^{\pm\infty} |n|^{\alpha} |K_{\pm}(n, n) - K_{\pm}(n \pm 1, n \pm 1)| < \infty,$$

(3.10)
$$\sum_{n=n_0}^{+\infty} |n|^{\alpha} |a^+(n)\kappa_+(n, n+1) - a^+(n-1)\kappa_+(n-1, n)| < \infty,$$

$$\sum_{n=n_0}^{-\infty} |n|^{\alpha} |a^-(n-1)\kappa_-(n, n-1) - a^-(n)\kappa_-(n+1, n)| < \infty,$$

where $\alpha = q$ for $q = 2, 3, \ldots$ and $\alpha = 0$ for $q = 1$.

Proof. It is almost literally the same as for Lemma 2.4, **(ii)**. For $n \geq n_0$, define

$$\sigma(n) = \sup_{j \geq n_0} (\hat{C}_+(j) + C_+(j)) \sum_{\ell=n}^{+\infty} p_+(\ell), \quad \sigma_1(n) := \sum_{\ell=n}^{+\infty} \sigma(\ell),$$

where $p_+(\ell)$, $\hat{C}_+(n)$, and $C_+(n)$ are the functions in (3.1) and (2.37). In the same way as we derived (2.49) we show

$$\sum_{n=n_0}^{+\infty} |n|^{\alpha} |a^+(n)\kappa_+(n, n+1) - a^+(n-1)\kappa_+(n-1, n)|$$

$$\leq \sum_{n=n_0}^{+\infty} |n|^{\alpha} |a^+(n)F_+(n, n+1) - a^+(n-1)F_+(n-1, n)|$$

$$+ 2 \sup_{j \in \mathbb{Z}} a^+(j) \times \begin{cases} \sum_{n=n_0}^{+\infty} n\sigma(2n) \sum_{\ell=n_0}^{+\infty} (n+\ell)^{q-1} \sigma(n+\ell), & q = 2, \ldots, \\ \sigma_1(2n_0)^2, & q = 1. \end{cases}$$

The remaining estimates in (3.10) are obtained similarly. $\qquad\square$

Note that (3.10) imply

(3.11)
$$n^{\alpha} (|a_{\pm}(n) - a^{\pm}(n)| + |b_{\pm}(n) - b^{\pm}(n)|) \in \ell^1(\mathbb{Z}_{\pm}).$$

Therefore, our sequences $a_+(n)$, $b_+(n)$ (resp. $a_-(n)$, $b_-(n)$) have the desired behavior as $n \to +\infty$ (resp. $n \to -\infty$), but nothing can be said about their behavior on the opposite half-axis.

Step 7. It remains to prove that $a_+(n) \equiv a_-(n)$ and $b_+(n) \equiv b_-(n)$ under conditions **I–III**, which are the same for all fixed $q = 1, 2, \ldots$. We already utilized conditions **I**, **(a)–(c)**, **II**, **III**, **(a)**, and $\mathbf{IV_q}$ in the proofs of Lemmas 3.1, 3.2 to obtain $a_{\pm}(n)$, $b_{\pm}(n)$ satisfying (3.11). The remaining conditions **III**, **(b)**, and **I**, **(d)**, play a key role for proving the uniqueness theorem. We remark that **III**, **(b)**, and (2.29), (2.30) are the analog of the Marchenko condition in the step-like finite-gap case.

Theorem 3.3. *The functions defined in (3.9) coincide, $a_+(n) \equiv a_-(n) = a(n)$, $b_+(n) \equiv b_-(n) = b(n)$. Moreover, the set $\mathcal{S}$, which satisfies conditions $\mathbf{I}$–$\mathbf{IV_q}$, is the set of scattering data for $H \in \mathcal{B}_{\alpha}(H^+, H^-)$ associated with the reconstructed coefficients $a(n)$, $b(n)$, where $\alpha = 0$ for $q = 1$ and $\alpha = q$ for $q = 2, 3, \ldots$.*

This theorem was proven in [15] by the Marchenko approach. In the remaining part of this section we briefly discuss how the direct/inverse scattering problem can be solved in the class $\mathcal{B}_1(H^+_{const}, H^-_{const})$, where $H^\pm_{const}$ are the Jacobi operators with constant coefficients defined in (1.4). So far, results are only known for $\mathcal{B}_2(H^+_{const}, H^-_{const})$, see [36]. For $\mathcal{B}_1(H^+_{const}, H^-_{const})$, the characteristic properties listed in Lemma 2.2 can be simplified. First of all, in this case the background Weyl solutions have no poles and hence $\hat{\delta}_\pm(\lambda)$, $\tilde{\delta}_\pm(\lambda)$ are obsolete. After the change of variables

$$(3.12) \qquad \lambda - b^\pm = a^\pm(z_\pm + z_\pm^{-1}),$$

condition **II** is given by

II$_{\mathbf{const}}$. *The functions $T_\pm(\lambda)$ can be continued analytically to $\mathbb{C} \setminus (\sigma \cup \sigma_d)$. They satisfy $T_+(\infty) = T_-(\infty) > 0$ and*

$$a^+(z_+ - z_+^{-1})T_+^{-1}(\lambda) = a^-(z_- - z_-^{-1})T_-^{-1}(\lambda) = W(\lambda).$$

The function $W(\lambda)$ is holomorphic in $\mathbb{C} \setminus \sigma$ and continuous up to the boundary. Moreover, $\overline{W(\lambda^{\mathrm{u}})} = W(\lambda^{\mathrm{l}})$ for $\lambda \in \sigma$ and $W(\lambda) \in \mathbb{R}$ for $\lambda \in \mathbb{R} \setminus \sigma$. It has simple zeros at the points λ_k, where $(W'(\lambda_k))^{-2} = \gamma_{+,k}\gamma_{-,k}$. It can have simple zeros on $\Sigma_v := \partial\sigma \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)})$, but does not vanish at other points of σ. If $W(E) = 0$ for $E \in \Sigma_v$, then

$$(3.13) \qquad W(\lambda) = C\sqrt{\lambda - E}(1 + o(1)), \quad C \neq 0, \quad \text{for } \lambda \to E.$$

Property (3.13) is new and was never noticed for perturbations with finite first moments. We will prove it in Section 4. As we see, the behavior of the Wronskian on constant background can be better controlled as on arbitrary finite-gap backgrounds. In fact, (3.13) shows that the Wronskian has the same behavior for first moments as for higher moments (see remarks to Lemma 2.2) in this case.

For constant backgrounds property **III** is replaced by

III$_{\mathbf{const}}$: (a) *The reflection coefficients $R_\pm(\lambda)$ are continuous functions on $\sigma_\pm^{\mathrm{u,l}}$;* (b) *If $E \in \partial\sigma_+ \cap \partial\sigma_-$ and $W(E) \neq 0$, then $R_\pm(E) = -1$.*

The difference between **III$_{\mathbf{const}}$** and **III** arises from the fact that $R_\pm(\lambda)$ are continuous at the points of Σ_v in the resonance case ($W(E) = 0$) too. For a proof of **III$_{\mathbf{const}}$** see Section 4.

Now the kernels of the Marchenko equations are given by $F_\pm(n, m) = F_\pm(n+m)$, where

$$F_\pm(n) = \frac{1}{2\pi \mathrm{i} a^\pm} \int_{|z_\pm|=1} R_\pm(\lambda) z_\pm^{\pm n}(z_\pm - z_\pm^{-1})^{-1} d\lambda$$

$$+ \frac{1}{2\pi \mathrm{i} a^\mp} \int_{\sigma_\mp^{(1)}} |T_\mp(\lambda)|^2 z_\pm^{\pm n}(z_\mp - z_\mp^{-1})^{-1} d\lambda + \sum_{k=1}^q \gamma_{\pm,k} z_{\pm,k}^{\pm n}.$$

The estimates on the transformation operator for constant backgrounds were announced in [19] and rigorously proven in [34, Ch. 10]. They allow to derive from the Marchenko equations that

$$(3.14) \qquad \sum_{n=1}^{\pm\infty} |n| |F_\pm(n \pm 2) - F_\pm(n)| < \infty.$$

It turns out (cf. [34]), that these estimates are sufficient to reconstruct a solution of the inverse scattering problem which belongs to $\mathcal{B}_1(H_{const}^+, H_{const}^-)$. In summary,

Theorem 3.4. *Conditions* **I**, **II_const**, **III_const**, *and* (3.14) *for the scattering data* (2.26) *are necessary and sufficient for solving the direct/inverse scattering problem in the class* $\mathcal{B}_1(H_{const}^+, H_{const}^-)$.

4. On the continuity of the reflection coefficients

The continuity of the reflection coefficient at the edge of the continuous spectrum in the resonance case has a long history. This question arose around 1975 in an attempt to clarify some characteristic properties, namely the Marchenko condition, for the scattering data of the Schrödinger equation on the whole axis with a fast decaying potential of finite first moment. For higher moments q starting from $q = 2$, the answer is evident. For such q, the Jost solutions and their conjugates are differentiable from the side of the spectrum with respect to the local parameter $\sqrt{\lambda - E}$, where E ($E = 0$ for the decaying potential) is the edge of the continuous spectrum. Therefore, their Wronskians are also differentiable with respect to the local parameter, and to prove continuity of the reflection coefficient in the resonance case it is sufficient to use de l'Hopital. This situation complicates a lot for finite first moments, because the Jost solutions are no longer differentiable at E. The problem for first moments was solved independently by Gusseinov [18] and Klaus [26] (for a revised version see [1]).

In this section, we combine methods developed in [18], [1] and adapt them to the discrete model to prove (3.13) and the continuity of the reflection coefficient first for the Jacobi operator $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ with real coefficients $a(n) > 0$, $b(n)$ satisfying

$$(4.1) \qquad \sum_{n \in \mathbb{Z}} |n| \left(\left| a(n) - \frac{1}{2} \right| + |b(n)| \right) < \infty.$$

This result is of independent interest. Then we prove (3.13) and **III_const**, (**a**) for arbitrary constant backgrounds which completes the proof of Theorem 3.4.

The Jacobi spectral equation with coefficients satisfying (4.1),

$$(4.2) \quad a(n)y(z, n+1) + a(n-1)y(z, n-1) + b(n)y(z, n) = \frac{1}{2}(z + z^{-1})y(z, n), \quad |z| \leq 1,$$

has the following two Jost solutions,

$$(4.3) \qquad \varphi_\pm(z, j) = \sum_{\ell=j}^{\pm\infty} K_\pm(j, \ell) z^{\pm\ell}.$$

Recall that the kernels $K_\pm(j, \ell)$ of the transformation operators satisfy

$$(4.4) \qquad |K_\pm(j, \ell)| \leq C \sum_{n=\left[\frac{j+\ell}{2}\right]}^{\pm\infty} \left(\left| a(n) - \frac{1}{2} \right| + |b(n)| \right)$$

in this case. The functions $C_\pm(\cdot)$ in (2.16) which decay in one direction and grow in the other direction are replaced by a constant C here (and in the general case of coinciding finite-gap backgrounds $H^+ = H^-$ as well). It follows from (4.1), (4.4)

that $\varphi_\pm(z)$ are continuous for $z \in \mathbb{T} := \{z : |z| = 1\}$. In general, they are not differentiable with respect to z at $\hat{z} = +1, -1$, since $\sum_{\ell=j}^{\pm\infty} |\ell K_\pm(j, \ell)|$ diverge. Let

$$W(z) = W(\varphi_-, \varphi_+) = a(0)(\varphi_-(z, 0)\varphi_+(z, 1) - \varphi_-(z, 1)\varphi_+(z, 0))$$

be the Wronskian of the Jost solutions and define

$$W_1(z) = a(0)(\varphi_-(z, 0)\overline{\varphi_+(z, 1)} - \varphi_-(z, 1)\overline{\varphi_+(z, 0)}).$$

Lemma 4.1. *For $\hat{z} \in \{1, -1\}$, suppose that $W(\hat{z}) = 0$. Then the following representations are valid*

(4.5) $$W(z) = C(z - \hat{z})(1 + o(1)) \quad as \ z \to \hat{z},$$

(4.6) $$W_1(z) = C_1(z - \hat{z})(1 + o(1)) \quad as \ z \to \hat{z} \ and \ z \in \mathbb{T},$$

where $C \neq 0$ and C_1 are constants.

Proof. We will use summation by parts, i.e., the following identity,

(4.7) $$\sum_{\ell=s}^{\pm\infty} \big(f(\ell) - f(\ell \pm 1)\big)v(\ell) = \sum_{\ell=s}^{\pm\infty} f(\ell)\big(v(\ell) - v(\ell \mp 1)\big) + f(s)v(s \mp 1),$$

which is valid for all $f(\cdot) \in \ell^1(\mathbb{Z}_\pm)$, $\sup_{\ell\in\mathbb{Z}_\pm} |v(\ell)| < \infty$ or vice versa.

Introduce

(4.8) $$\Phi_\pm^{(j)}(s) = \sum_{\ell=s}^{\pm\infty} K_\pm(j, \ell)\hat{z}^\ell$$

which are well defined due to (4.4) and (4.1). Moreover, since $\hat{z}^{-1} = \hat{z}$,

(4.9) $$\Phi_\pm^{(j)}(j) = \varphi_\pm(\hat{z}, j).$$

Applying (4.7) to (4.3) we obtain

$$\varphi_\pm(z, j) = \sum_{\ell=j}^{\pm\infty} \Phi_\pm^{(j)}(\ell)\big((\hat{z}z)^{\pm\ell} - (\hat{z}z)^{\pm\ell-1}\big) + \Phi_\pm^{(j)}(j)(\hat{z}z)^{\pm j-1}$$

$$= \varphi_\pm(\hat{z}, j) + \Phi_\pm^{(j)}(j)\left((\hat{z}z)^{\pm j-1} - \hat{z}^{\pm 2j-2}\right) + \sum_{\ell=j}^{\pm\infty} \Phi_\pm^{(j)}(\ell)(\hat{z}z)^{\pm l}\left(1 - (\hat{z}z)^{-1}\right).$$

Abbreviate

$$\zeta(z) := \frac{z - \hat{z}}{z},$$

then

(4.10) $$\varphi_\pm(z, 0) = \zeta(z) \sum_{\ell=\frac{1\pm1}{2}}^{\pm\infty} \Phi_\pm^{(0)}(\ell)(\hat{z}z)^{\pm\ell} + (z\hat{z})^{\frac{\pm1-1}{2}}\varphi_\pm(\hat{z}, 0),$$

(4.11) $$\varphi_\pm(z, 1) = \zeta(z) \sum_{\ell=\frac{1\pm1}{2}}^{\pm\infty} \Phi_\pm^{(1)}(\ell)(\hat{z}z)^{\pm\ell} + (z\hat{z})^{\frac{\pm1-1}{2}}\varphi_\pm(\hat{z}, 1).$$

Multiplying (4.10) by $\varphi_\pm(\hat{z}, 1)$ and (4.11) by $\varphi_\pm(\hat{z}, 0)$, their difference is equal to

(4.12) $$\breve{W}_\pm(z) := \varphi_\pm(z, 1)\varphi_\pm(\hat{z}, 0) - \varphi_\pm(z, 0)\varphi_\pm(\hat{z}, 1) = \zeta(z)\Psi_\pm(z),$$

where

$$\Psi_\pm(z) := \sum_{\ell=\frac{1\pm1}{2}}^{\pm\infty} h_\pm(\ell)(\hat{z}z)^{\pm\ell},$$

$$(4.13) \qquad h_\pm(\ell) := \Phi_\pm^{(1)}(\ell)\varphi_\pm(\hat{z},0) - \Phi_\pm^{(0)}(\ell)\varphi_\pm(\hat{z},1).$$

Note that by (4.8), (4.4) we have $h_\pm(\cdot) \in \ell^\infty(\mathbb{Z}_\pm)$. Suppose that we can show

$$(4.14) \qquad h_\pm(\cdot) \in \ell^1(\mathbb{Z}_\pm).$$

Then $\Psi_\pm(z)$ are continuous at $\hat{z}$ and (4.12) implies

$$(4.15) \qquad \frac{\varphi_\pm(z,1)}{\varphi_\pm(z,0)} = \frac{\varphi_\pm(\hat{z},1)}{\varphi_\pm(\hat{z},0)} + C_0^\pm(z-\hat{z})(1+o(1)), \quad \text{as } \varphi_\pm(\hat{z},0) \neq 0,$$

$$(4.16) \qquad \frac{\varphi_\pm(z,0)}{\varphi_\pm(z,1)} = \frac{\varphi_\pm(\hat{z},0)}{\varphi_\pm(\hat{z},1)} + C_1^\pm(z-\hat{z})(1+o(1)), \quad \text{as } \varphi_\pm(\hat{z},1) \neq 0,$$

for constants $C_1^\pm$, $C_0^\pm$. Both representations hold for all $|z| \leq 1$ in a vicinity of $\hat{z}$.

Hence, if $W(\hat{z}) = 0$, then $\varphi_-(\hat{z},j) = \gamma\varphi_+(\hat{z},j)$ and we have two possibilities, either

$$(4.17) \qquad \varphi_-(\hat{z},0)\varphi_+(\hat{z},0) \neq 0,$$

or

$$(4.18) \qquad \varphi_-(\hat{z},0) = \varphi_+(\hat{z},0) = 0.$$

In the first case we obtain from (4.15)

$$W(z) = a(0)\varphi_-(z,0)\varphi_+(z,0)\Big(\frac{\varphi_+(\hat{z},1)}{\varphi_+(\hat{z},0)} - \frac{\varphi_-(\hat{z},1)}{\varphi_-(\hat{z},0)} + (C_0^+ - C_0^-)(z-\hat{z})(1+o(1))\Big)$$

$$(4.19)$$
$$= a(0)\varphi_-(\hat{z},0)\varphi_+(\hat{z},0)(C_0^+ - C_0^-)(z-\hat{z})(1+o(1)),$$

which implies (4.5) with $C = a(0)\varphi_-(\hat{z},0)\varphi_+(\hat{z},0)(C_0^+ - C_0^-)$. It follows from the general estimate (2.29) that $(z^2-1)/W(z)$ is bounded for $z \in \mathbb{T}$ in small vicinities of $+1, -1$, therefore $C \neq 0$. For the case (4.18) we use (4.16). To obtain (4.6), we substitute z^{-1} for z (which is possible as $z \in \mathbb{T}$) in the "+" cases of (4.3), (4.9)–(4.16) and also use $\overline{\hat{z}} = \hat{z}$.

Therefore, in order to finish the proof it remains to show (4.14). Let us first consider h_+. For $m \geq 1$, the transformation operators involved in (4.14) satisfy the following Marchenko equations,

$$K_+(0,m) + \sum_{\ell=0}^{+\infty} K_+(0,\ell)F_+(\ell+m) = 0,$$

$$K_+(1,m) + \sum_{\ell=1}^{+\infty} K_+(1,\ell)F_+(\ell+m) = \frac{\delta(1,m)}{K_+(1,1)}.$$

Multiplying both equalities by $\hat{z}^m$ and summing from $m = s \geq 1$ to $+\infty$ gives

$$\Phi_+^{(0)}(s) + \sum_{m=s}^{+\infty} \sum_{\ell=0}^{+\infty} F_+(\ell + m)\hat{z}^{\ell+m} \left(\Phi_+^{(0)}(\ell) - \Phi_+^{(0)}(\ell+1) \right) = 0,$$

$$\Phi_+^{(1)}(s) + \sum_{m=s}^{+\infty} \sum_{\ell=1}^{+\infty} F_+(\ell + m)\hat{z}^{\ell+m} \left(\Phi_+^{(1)}(\ell) - \Phi_+^{(1)}(\ell+1) \right) = \frac{\delta(1,s)}{K_+(1,1)}.$$

We set $v(\ell) = F_+(\ell)\hat{z}^\ell$ and sum by parts according to (4.7),

$$\Phi_+^{(0)}(s) + \sum_{m=s}^{+\infty} \left(\sum_{\ell=0}^{+\infty} (v(\ell+m) - v(\ell+m-1)) \Phi_+^{(0)}(\ell) + \Phi_+^{(0)}(0)v(m-1) \right) = 0,$$

$$\Phi_+^{(1)}(s) + \sum_{m=s}^{+\infty} \left(\sum_{\ell=1}^{+\infty} (v(\ell+m) - v(\ell+m-1)) \Phi_+^{(1)}(\ell) + \Phi_+^{(1)}(1)v(m) \right) = \frac{\delta(1,s)}{K_+(1,1)}.$$

Summing over m and taking (4.9) into account yields

$$(4.20) \qquad \Phi_+^{(0)}(s) + \varphi_+(\hat{z},0) \sum_{m=s}^{+\infty} v(m) - \sum_{\ell=1}^{+\infty} \Phi_+^{(0)}(\ell)v(\ell+s-1) = 0,$$

$$\Phi_+^{(1)}(s) + \varphi_+(\hat{z},1) \sum_{m=s}^{+\infty} v(m) - \sum_{\ell=1}^{+\infty} \Phi_+^{(1)}(\ell)v(\ell+s-1) = \frac{\delta(1,s)}{K_+(1,1)}.$$

We again have to distinguish two cases. If (4.17) holds, we multiply the first equation by $\varphi_+(\hat{z},1)$, the second by $\varphi_+(\hat{z},0)$, subtract the first from the second, and use (4.13) to arrive at

$$(4.21) \qquad h_+(s) - \sum_{\ell=1}^{+\infty} h_+(\ell)v(\ell+s-1) = \frac{\delta(1,s)}{K_+(1,1)}\varphi_+(\hat{z},0).$$

We know that $h_+(\cdot) \in \ell^\infty(\mathbb{Z}_+)$, and from the estimates given in [19] and [34, Ch. 10] we have (recall that $|v(\ell)| = |F_+(\ell)|$)

$$(4.22) \qquad |v(\ell)| \leq C \sum_{n=\left[\frac{\ell}{2}\right]}^{+\infty} \left(\left| a(n) - \frac{1}{2} \right| + |b(n)| \right).$$

But any bounded solution of (4.21) with a kernel satisfying (4.22) already is in $\ell^1(\mathbb{Z}_+)$, as proved in [30] (see (3.2.24)–(3.2.25)). For the case (4.18) we have

$$\varphi_+(\hat{z},0) = 0, \quad h_+(s) = \varphi_+(\hat{z},1)\Phi_+^{(0)}(s).$$

Therefore, (4.20) implies

$$h_+(s) - \sum_{\ell=1}^{+\infty} h_+(\ell)v(\ell+s-1) = 0,$$

and hence again $h_+(\cdot) \in \ell^1(\mathbb{Z}_+)$. $\qquad\square$

The continuity of the reflection coefficients $R_\pm$ follows from Lemma 4.1, since $R_+(z) = -W_1(z)W^{-1}(z)$, $R_-(z) = \overline{W_1(z)}W^{-1}(z)$.

Now we are ready to prove (3.13) for the case of two different constant background operators $H_{const}^{\pm}$, defined by (1.4), and $q = 1$. The spectra $\sigma_{\pm}$ of $H_{const}^{\pm}$ consist of intervals,

$$(4.23) \qquad \sigma_{\pm} = [c^{\pm}, d^{\pm}], \quad \text{where } c^{\pm} := b^{\pm} - 2a^{\pm}, \, d^{\pm} := b^{\pm} + 2a^{\pm}.$$

Let us first specify the set Σ_v of possible virtual levels for this case. In order to complete the proof of property $\mathbf{III_{const}}$, $(\mathbf{a})$, compared to $\mathbf{III}$, $(\mathbf{a})$, we have to show continuity of one or the other reflection coefficient on Σ_v. The following mutual locations of the spectra $\sigma_{\pm}$ are possible.

   (i) $\sigma_- \subset \sigma_+$ (or vice-versa). Then $\Sigma_v = \partial\sigma_+$, and we have to show continuity of R_+;
  (ii) $d^+ \leq c^-$ (or $d^- \leq c^+$). Then $\Sigma_v = \{c^+, d^+, c^-, d^-\}$, and we have to show continuity of $R_{\pm}$ at $c^{\pm}$ and $d^{\pm}$;
 (iii) $\sigma^{(2)} \neq \emptyset$, for example, $\sigma^{(2)} = [c^-, d^+]$. Then $\Sigma_v = \{c^+, d^-\}$, and we have to show continuity of R_+ at c^+ and of R_- at d^-.

We start with case (i), $\Sigma_v = \{c^+, d^+\}$. Suppose that $E = d^+$, $W(E) = 0$, and introduce the change of variables $\lambda - b^+ = a^+(z + z^{-1})$ (cf. (3.12)). The point d^+ is mapped to $\hat{z} = 1$ and the Jost solution $\phi_+(\lambda) = \varphi_+(z)$ is now considered as a function of z on the unit circle with representation (4.3). Repeating literally the beginning of the proof of Lemma 4.1 we obtain (4.12) and (4.13) for $\varphi_+(z)$. Then (4.15) and (4.16) follow from (4.14). Observe that the only structural information on the kernel of the Marchenko equation which was used to prove (4.14) is inequality (4.22). But (4.22) holds in the steplike case too, since we use it for $\ell \geq 0$ where $C_1(n)$ (see (2.45), (2.16)) can be replaced by a constant.

Hence (4.15) and (4.16) are valid in the steplike case, but only for the Jost solution for which the point $\hat{z}$ is an edge of the spectrum of its background operator. In turn, (4.15) and (4.16) imply for all λ in a vicinity of $E = d^+$,

$$(4.24) \qquad \frac{\phi_+(\lambda, 1)}{\phi_+(\lambda, 0)} = \frac{\phi_+(E, 1)}{\phi_+(E, 0)} + C_0^+ \sqrt{\lambda - E}(1 + o(1)), \quad \text{as } \phi_+(E, 0) \neq 0,$$

$$(4.25) \qquad \frac{\phi_+(\lambda, 0)}{\phi_+(\lambda, 1)} = \frac{\phi_+(E, 0)}{\phi_+(E, 1)} + C_1^+ \sqrt{\lambda - E}(1 + o(1)), \quad \text{as } \phi_+(E, 1) \neq 0,$$

where C_1^+, C_0^+ are constants. Evidently, the same expressions will hold for $\overline{\frac{\phi_+(\lambda, 1)}{\phi_+(\lambda, 0)}}$ (resp. $\overline{\frac{\phi_+(\lambda, 0)}{\phi_+(\lambda, 1)}}$) as $\lambda \to E$ and $\lambda < E$.

On the other hand, if $d^- < d^+$ then $\phi_-(\lambda, \cdot)$ are C^{∞}-functions of λ, and we have

$$(4.26) \qquad \frac{\phi_-(\lambda, 1)}{\phi_-(\lambda, 0)} = \frac{\phi_-(E, 1)}{\phi_-(E, 0)} + \hat{C}_0^-(\lambda - E)(1 + o(1)), \quad \text{as } \phi_-(E, 0) \neq 0,$$

$$(4.27) \qquad \frac{\phi_-(\lambda, 0)}{\phi_-(\lambda, 1)} = \frac{\phi_-(E, 0)}{\phi_-(E, 1)} + \hat{C}_1^-(\lambda - E)(1 + o(1)), \quad \text{as } \phi_-(E, 1) \neq 0,$$

for constants $\hat{C}_1^-$, $\hat{C}_0^-$. If $d^+ = d^-$, then upon the change of variables $\lambda - b^- = a^-(z + z^{-1})$, where $d^- = d^+ = E$ corresponds to $\hat{z} = 1$, we obtain (4.15), (4.16) for

$\varphi_-(z, \cdot)$. Therefore,

$$(4.28) \qquad \frac{\phi_-(\lambda, 1)}{\phi_-(\lambda, 0)} = \frac{\phi_-(E, 1)}{\phi_-(E, 0)} + C_0^- \sqrt{\lambda - E}(1 + o(1)), \quad \text{as } \phi_-(E, 0) \neq 0,$$

$$(4.29) \qquad \frac{\phi_-(\lambda, 0)}{\phi_-(\lambda, 1)} = \frac{\phi_-(E, 0)}{\phi_-(E, 1)} + C_1^- \sqrt{\lambda - E}(1 + o(1)), \quad \text{as } \phi_-(E, 1) \neq 0,$$

for all λ in a small vicinity of E. The same considerations as for (4.19) imply now for the case (4.17) that $W(\lambda) = C\sqrt{\lambda - E}(1+o(1))$ with $C = -a(0)\phi_-(\lambda, 0)\phi_+(\lambda, 0)C_0^+$ if $d^- < d^+$, and $C = a(0)\phi_-(\lambda, 0)\phi_+(\lambda, 0)(C_0^- - C_0^+)$ if $d^- = d^+$. Due to (2.29), $C \neq 0$, which proves (3.13) in this case. It is clear that for

$$W_1(\lambda) = a(0)(\phi_-(\lambda, 0)\overline{\phi_+(\lambda, 1)} - \phi_-(\lambda, 1)\overline{\phi_+(\lambda, 0)}$$

the representation $W_1(\lambda) = C_1\sqrt{\lambda - E}(1+o(1))$ from the side of the spectrum will hold, but here we cannot check whether C_1 vanishes or not. This representation and (3.13) imply the continuity of the reflection coefficient R_+ for case (i). Cases (ii) and (iii) can be treated in the same way, which finishes the proof of Theorem 3.4.

References

[1] T. Aktosun and M. Klaus, *Small energy asymptotics for the Schrödinger equation on the line*, Inverse problems **17**, 619–632 (2001).

[2] J. Bazargan and I. Egorova, *Jacobi operator with step-like asymptotically periodic coefficients*, Mat. Fiz. Anal. Geom. **10**, 425–442 (2003).

[3] A. Boutet de Monvel and I. Egorova, *The Toda lattice with step-like initial data. Soliton asymptotics*, Inverse Problems **16**, 955–977 (2000).

[4] A. Boutet de Monvel, I. Egorova, and E. Khruslov, *Soliton asymptotics of the Cauchy problem solution for the Toda lattice*, Inverse Problems **13**, No. 2, 223–237 (1997).

[5] A. Boutet de Monvel, I. Egorova, and G. Teschl, *Inverse scattering theory for one-dimensional Schrödinger operators with steplike finite-gap potentials*, J. Anal. Math. **106**, 271–316 (2008).

[6] K.M. Case and M. Kac, *A discrete version of the inverse scattering problem*, J. Math. Phys. **14**, 594–603 (1973).

[7] K.M. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys. **14**, 2166–2175 (1973).

[8] K.M. Case and J. Geronimo, *Scattering theory and polynomials orthogonal on the real line*, Trans. Amer. Math. Soc. **258**, 467–494 (1980).

[9] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. **49**, 35–83 (1996).

[10] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32**, 121–251 (1979).

[11] I. Egorova, *The scattering problem for step-like Jacobi operator*, Mat. Fiz. Anal. Geom. **9**, 188–205 (2002).

[12] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with quasi-periodic background*, Comm. Math. Phys. **264**, 811–842 (2006).

[13] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with steplike quasi-periodic background*, Inverse Problems **23**, 905–918 (2007).

[14] I. Egorova, J. Michor, and G. Teschl, *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proc. Amer. Math. Soc. **135**, 1817–1827 (2007).

[15] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with general steplike quasi-periodic background*, Zh. Mat. Fiz. Anal. Geom. **4**, 33–62 (2008).

[16] I. Egorova, J. Michor, and G. Teschl, *Soliton solutions of the Toda hierarchy on quasi-periodic backgrounds revisited*, Math. Nach. **282**, 526–539 (2009).

[17] L.D. Faddeev, *Properties of the S-matrix of the one-dimensional Schrödinger equation*, Trudy Mat. Inst. Steklov. **73**, 314–336 (1964). (Russian)

[18] I.M. Gusseinov, *On the continuity of the reflection coefficient for one-dimensional Schrödinger equation*, Diff. Equations **21**, 1993–1995 (1985).

[19] G.S. Guseinov, *The inverse problem of scattering theory for a second-order difference equation on the whole axis*, Soviet Math. Dokl. **17**, 1684–1688 (1976).

[20] S. Kamvissis and G. Teschl, *Long-time asymptotics of the periodic Toda lattice under short-range perturbations*, Phys. Lett. A, **364**, 480–483 (2007).

[21] S. Kamvissis and G. Teschl, *Stability of the periodic Toda lattice under short range perturbations*, J. Math. Phys. **53**, 073706 (2012).

[22] J. Kay and H. Moses, *The determination of the scattering potential from the spectral measure function I-III*, Nuovo Cimento **2**, 917–961 (1955); **3**, 56–84 (1956); **3**, 276–304 (1956).

[23] A.K. Khanmamedov, *Transformation operators for the perturbed Hill difference equation and one of their applications*, Sibirsk. Mat. Zh. **44**, 926–937 (2003). (Russian)

[24] A.K. Khanmamedov, *Direct and inverse scattering problems for the perturbed Hill difference equation*, Sb. Math. **196**, 1529–1552 (2005).

[25] A.K. Khanmamedov, *On the continuity of the reflection coefficient for difference Schrödinger operator with divergent potential*, Baku University Bulletin. Mathematics and Physics **2**, 54–58 (2005). (Russian)

[26] M. Klaus, *Low-energy behaviour of the scattering matrix for the Schrödinger equation on the line*, Inverse Problems **4**, 505–512 (1988).

[27] H. Krüger and G. Teschl, *Long-time asymptotics of the Toda lattice for decaying initial data revisited*, Rev. Math. Phys. **21**, 61–109 (2009).

[28] H. Krüger and G. Teschl, *Stability of the periodic Toda lattice in the soliton region*, Int. Math. Res. Not. **2009**, 3996–4031 (2009).

[29] V.A. Marchenko, *On reconstruction of the potential energy from phases of the scattered waves*, Dokl. Akad. Nauk SSSR **104**, 695–698 (1955). (Russian)

[30] V.A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[31] J. Michor and G. Teschl, *Trace formulas for Jacobi operators in connection with scattering theory for quasi-periodic background*, Operator Theory, Analysis, and Mathematical Physics, J. Janas, et al. (eds.), 69–76, Oper. Theory Adv. Appl., 174, Birkhäuser, Basel, 2007.

[32] G. Teschl *Oscillation theory and renormalized oscillation theory for Jacobi operators*, J. Diff. Eqs. **129**, 532–558 (1996).

[33] G. Teschl, *On the initial value problem for the Toda and Kac-van Moerbeke hierarchies*, in Differential Equations and Mathematical Physics, R. Weikard and G. Weinstein (eds.), 375–384, AMS/IP Studies in Advanced Mathematics 16, Amer. Math. Soc., Providence, 2000.

[34] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.

[35] E.C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Vol. 2, Clarendon Press, Oxford, 1958.

[36] S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Comm. Pure Appl. Math. **44**, 1171–1242 (1991).

[37] A. Volberg and P. Yuditskii, *On the inverse scattering problem for Jacobi matrices with the spectrum on an interval, a finite system of intervals or a Cantor set of positive length*, Comm. Math. Phys. **226**, 567–605 (2002).

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