INVERSE SCATTERING THEORY FOR ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH STEPLIKE FINITE-GAP POTENTIALS

ANNE BOUTET DE MONVEL, IRYNA EGOROVA, AND GERALD TESCHL

Abstract. We develop direct and inverse scattering theory for one-dimensional Schrödinger operators with steplike potentials which are asymptotically close to different finite-gap potentials on different half-axes. We give a complete characterization of the scattering data, which allow unique solvability of the inverse scattering problem in the class of perturbations with finite second moment.

1. Introduction

In this paper we consider direct and inverse scattering theory for one-dimensional Schrödinger operators with steplike finite-gap background, using the approach by Marchenko [26].

To set the stage, let

$$H_{\pm} = -\frac{d^2}{dx^2} + p_{\pm}(x), \quad x \in \mathbb{R}, \tag{1.1}$$

be two (in general different) one-dimensional Schrödinger operators with real finite-gap potentials $p_{\pm}(x)$. Furthermore, let

$$H = -\frac{d^2}{dx^2} + q(x), \quad x \in \mathbb{R}, \tag{1.2}$$

be the "perturbed" operator with real potential $q(x) \in L^1_{\text{loc}}$ such that

$$\pm \int_0^{\pm\infty} |q(x) - p_{\pm}(x)|(1 + x^2)dx < \infty. \tag{1.3}$$

That is, the potential $q(x)$ has different asymptotic behavior on different half-axes, and we will call it a *steplike* potential by analogy with the case of two different constant backgrounds.

The scattering problem for the operators (1.2)–(1.3) is classical and arises in various physical applications, for example, when studying properties of the alloy of two different semi-infinite one dimensional crystals. We refer to the recent work [18] for a more detailed discussion of the history of such problems together with further references to the literature. In addition, this scattering problem is of course important for the solution of the Korteweg–de Vries (KdV) equation with initial data in these classes.

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For a constant steplike background, that is, $p_\pm(x) = c_\pm$ ($c_+ \neq c_-$ some constants), this problem was completely solved in [2, 6, 7], including applications to the initial value problem for the KdV equation ([5, 20]), and the asymptotic behavior of the solution for large time ([22]). The case, when the potential vanishes on the right half axis ($p_+(x) = 0$) and is asymptotically periodic finite band on the left half axis, was considered in [11, 30, 31, 33]. The initial value problem for the KdV equation was also solved in this case ([12]), and long time asymptotics can be found in [12, 23, 24] (see also [3] for the Jacobi operator case). The case of one-periodic background $p_-(x) = p_+(x)$ was studied in [13, 14, 15] and [28, 29] (for the Jacobi operators the same problem was considered in [8] with the extension to different background operators in the same isospectral class given in [10]).

However, despite the fact that several special cases are well understood by now, only very little was known about the general situation considered here. In fact, the various mutual locations of the respective background spectra (cf. the example on page 7 below) and background Dirichlet eigenvalues, produce a multitude of different cases. To illustrate this, we mention that only a classification of all possible singularities of the transmission coefficient at the boundary of the spectra would require 16 case distinctions (this is for example the reason why we formulate its properties in terms of the Wronskian of the Jost solutions in Lemma 3.3 **II** below). Our goal here is to find a complete characterization of the scattering data for the operator H, that will allow us to solve the inverse scattering problem and to prove the uniqueness of the reconstructed potential in the class (1.3) with the second moment finite. In particular, we will do this *without* any restrictions on the mutual location of the respective spectra and *without* any restrictions on the location of the Dirichlet eigenvalues. In this respect, note that for example in [13, 14, 15] it is required that the Dirichlet eigenvalues do not coincide with the edges of the continuous spectrum. In fact, it was demonstrated in [8] (for the case of Jacobi operators) that these cases give rise to a different behaviour of the scattering data, which does not occur in the constant background case. Furthermore, the inclusion (and understanding) of this case is important for applications to the solution of the initial value problem of KdV equation, since these cases are unavoidable under the KdV flow. We refer to [9] for a detailed discussion (in case of the Toda lattice) and we will give a brief outline for the KdV equation in Section 6.

On the other hand, we should also mention that there are two things which we do not address here: First of all, one could relax our decay assumption and replace the second moment in (1.3) by the first moment. Our approach is crafted in such a way that this causes no principal problems. To keep our presentation more readable we have decidedd not to include this case at this point. Secondly, one could allow general periodic potentials, that is, an infinite number of gaps. Again there are no serious impediments to treating this case.

Finally, let us give a brief outline of the present paper. We start with some preliminary notations and list some standard facts of the spectral analysis for the background Hill operators in Section 2. Then we study the properties of the scattering matrix for steplike operator, paying particular attention to analytical properties of its entries at the edges of the continuous spectrum of operator H (Section 3). In Section 4 we derive the Gel'fand–Levitan–Marchenko (GLM) equations and obtain complementary estimates on their kernels (see also Appendix A). In this section we also formulate our main result, that characterize the scattering data (Theorem 4.3).

Then we discuss the unique solvability of the GLM equations, that allows us to solve the inverse scattering problem. Section 5 is the most important section of the present paper. Here we discuss the scheme of the solution of the inverse scattering problem and prove the uniqueness of the reconstructed potential (Theorem 5.3). Our approach is modeled after the generalized Marchenko approach, developed in [26]. Our final Section 6 contains some applications to the KdV equation.

2. The Weyl solutions of the background operators

Let $H_\pm$ be two finite-gap one-dimensional Schrödinger operators associated with the potentials $p_\pm(x)$[1]. Let $s_\pm(z,x)$, $c_\pm(z,x)$ be sin- and cos-type solutions of the equation

$$\left(-\frac{d^2}{dx^2}+p_\pm(x)\right)y(x)=z\,y(x),\quad z\in\mathbb{C}, \tag{2.1}$$

associated with the initial conditions

$$s_\pm(z,0)=c'_\pm(z,0)=0,\quad c_\pm(z,0)=s'_\pm(z,0)=1, \tag{2.2}$$

where the prime denotes the derivative with respect to x.

It is well-known that finite-gap Schrödinger operators are associated with the Riemann surface of a square root of the type

$$\sqrt{-\prod_{j=0}^{2r_\pm}(z-E_j^\pm)},\qquad E_0^\pm<E_1^\pm<\cdots<E_{2r_\pm}^\pm, \tag{2.3}$$

where $r_\pm\in\mathbb{N}$. Moreover, $H_\pm$ are uniquely determined by fixing a Dirichlet divisor $\sum_{j=1}^{r^\pm}(\mu_j^\pm,\sigma_j^\pm)$, where $\mu_j^\pm\in[E_{2j-1}^\pm,E_{2j}^\pm]$ and $\sigma_j^\pm\in\{-1,1\}$. We refer the interested reader to [1, 17, 19, 25] for relevant background information. The reader not familiar with this theory can always think of the special case of periodic finite gap operators.

The spectra of $H_\pm$ consist of $r_\pm+1$ bands

$$\sigma_\pm:=[E_0^\pm,E_1^\pm]\cup\cdots\cup[E_{2j-2}^\pm,E_{2j-1}^\pm]\cup\cdots\cup[E_{2r_\pm}^\pm,\infty). \tag{2.4}$$

Note that in the special case where $p_\pm$ is periodic, we have merged all colliding bands. Let

$$M_{r_\pm}:=\left\{\mu_1^\pm,\dots,\mu_{r_\pm}^\pm\right\}$$

be the set of Dirichlet eigenvalues and set

$$g_\pm(z)=\frac{\prod_{j=1}^{r_\pm}(z-\mu_j^\pm)}{2\sqrt{-\prod_{j=0}^{2r_\pm}(z-E_j^\pm)}}, \tag{2.5}$$

where the branch of the square root is chosen such that we obtain a Herglotz–Nevanlinna function,

$$\operatorname{Im}(g_\pm(z))>0\quad\text{for}\quad\operatorname{Im}(z)>0. \tag{2.6}$$

[1] Everywhere in this paper the sub or super index "+" (resp. "−") refers to the background on the right (resp. left) half-axis.

Let us cut the complex plane along the spectrum $\sigma_\pm$ and denote the upper and lower sides of the cuts by $\sigma_\pm^{\mathrm{u}}$ and $\sigma_\pm^{\mathrm{l}}$. The corresponding points on these cuts will be denoted by λ^{u} and λ^{l}, respectively. In particular, this means

$$f(\lambda^{\mathrm{u}}) := \lim_{\varepsilon\downarrow 0} f(\lambda + \mathrm{i}\varepsilon), \qquad f(\lambda^{\mathrm{l}}) := \lim_{\varepsilon\downarrow 0} f(\lambda - \mathrm{i}\varepsilon), \qquad \lambda\in\sigma_\pm.$$

Condition (2.6) then implies

$$\frac{1}{\mathrm{i}} g_\pm(\lambda^{\mathrm{u}}) = \mathrm{Im}(g_\pm(\lambda^{\mathrm{u}})) > 0 \quad \text{for} \quad \lambda\in\sigma_\pm. \tag{2.7}$$

Next consider the Weyl solutions $\psi_\pm(z,x)$ and $\breve{\psi}_\pm(z,x)$ of (2.1) which are determined up to a multiplication constant, depending on z, by the requirement

$$\begin{aligned} &\psi_\pm(z,\,\cdot\,) \in L^2(\mathbb{R}_\pm),\\ &\text{resp. } \breve{\psi}_\pm(z,\,\cdot\,) \in L^2(\mathbb{R}_\mp) \end{aligned} \tag{2.8}$$

for $z\in\mathbb{C}\backslash\sigma_\pm$. We will normalize them according to $\psi_\pm(z,0) = \breve{\psi}_\pm(z,0) = 1$ such that

$$\begin{aligned} &\psi_\pm(z,x) = c_\pm(z,x) + m_\pm(z) s_\pm(z,x),\\ &\text{resp. } \breve{\psi}_\pm(z,x) = c_\pm(z,x) + \breve{m}_\pm(z) s_\pm(z,x), \end{aligned} \tag{2.9}$$

where

$$m_\pm(z) = \frac{\psi_\pm'(z,0)}{\psi_\pm(z,0)}, \qquad \breve{m}_\pm(z) = \frac{\breve{\psi}_\pm'(z,0)}{\breve{\psi}_\pm(z,0)}, \tag{2.10}$$

are the Weyl m-functions. In the case of periodic operators, $\psi_\pm(z,x)$ and $\breve{\psi}_\pm(z,x)$ are of course just the Floquet solutions. They are equal to the branches on the upper/lower sheet of the Baker-Akhiezer function of $H_\pm$.

It is well-known (see, for example, [25]), that $m_\pm(z) - \breve{m}_\pm(z) = \mp g_\pm(z)^{-1}$. Equations (2.2) and (2.10) then imply that the Wronskian of the Weyl solutions is equal to

$$W(\breve{\psi}_\pm(z), \psi_\pm(z)) = \mp g_\pm(z)^{-1}. \tag{2.11}$$

where $W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$ denotes the usual Wronski determinant.

The set of band edges is given by

$$\partial\sigma_\pm = \left\{E_0^\pm, E_1^\pm, \dots, E_{2r_\pm}^\pm\right\}. \tag{2.12}$$

For every Dirichlet eigenvalue $\mu_j^\pm$ the Weyl functions might have poles. If $\mu_j^\pm$ is in the interior of its gap, precisely one Weyl function $m_\pm$ or $\breve{m}_\pm$ will have a simple pole. Otherwise, if $\mu_j^\pm$ sits at an edge, both will have a square root singularity. Hence we divide the set of poles accordingly:

$$\begin{aligned} M_\pm &= \{\mu_j^\pm \mid \mu_j^\pm \in (E_{2j-1}, E_{2j}) \text{ and } m_\pm \text{ has a simple pole}\},\\ \breve{M}_\pm &= \{\mu_j^\pm \mid \mu_j^\pm \in (E_{2j-1}, E_{2j}) \text{ and } \breve{m}_\pm \text{ has a simple pole}\},\\ \hat{M}_\pm &= \{\mu_j^\pm \mid \mu_j^\pm \in \{E_{2j-1}, E_{2j}\}\}. \end{aligned}$$

Clearly $M_{r_\pm} = M_\pm \cup \breve{M}_\pm \cup \hat{M}_\pm$. Then we have

$$m_\pm(z) = \frac{C_\pm}{z-\mu}\left(1 + o(1)\right), \quad \breve{m}_\pm(z) = O(1),$$

for $z \to \mu \in M_\pm$,

$$m_\pm(z) = O(1), \quad \breve m_\pm(z) = \frac{\breve C_\pm}{z-\mu}\left(1+o(1)\right),$$

for $z \to \mu \in \breve M_\pm$, and

$$m_\pm(z) = \frac{C_\pm}{\sqrt{z-E}}\left(1+o(1)\right), \quad \breve m_\pm(z) = -\frac{C_\pm}{\sqrt{z-E}}\left(1+o(1)\right),$$

for $z \to E \in \hat M_\pm$. Here $C_\pm$,$\breve C_\pm$ denote some nonzero constants.

In particular, we obtain the following properties of the Weyl solutions (see, e.g., [17, 19, 25, 26, 34]):

Lemma 2.1. *The Weyl solutions have the following properties:*

(i) *The function $\psi_\pm(z,x)$ (resp. $\breve\psi_\pm(z,x)$) is holomorphic as a function of z in the domain $\mathbb{C}\setminus(\sigma_\pm \cup M_\pm)$ (resp. $\mathbb{C}\setminus(\sigma_\pm\cup\breve M_\pm)$), takes real values on the set $\mathbb{R}\setminus\sigma_\pm$, has simple poles at the points of the set $M_\pm$ (resp., $\breve M_\pm$). It is continuous up to the boundary $\sigma_\pm^{\mathrm u}\cup\sigma_\pm^{\mathrm l}$ except at the points from $\hat M_\pm$ and*

$$\psi_\pm(\lambda^{\mathrm u}) = \breve\psi_\pm(\lambda^{\mathrm l}) = \overline{\psi_\pm(\lambda^{\mathrm l})}, \quad \lambda\in\sigma_\pm. \tag{2.13}$$

For $E\in\hat M_\pm$ the Weyl solutions satisfy

$$\psi_\pm(z,x) = O\left(\frac{1}{\sqrt{z-E}}\right), \quad \breve\psi_\pm(z,x) = O\left(\frac{1}{\sqrt{z-E}}\right), \quad \text{as } z\to E\in\hat M_\pm.$$

The same is true for $\psi'_\pm(z,x)$ and $\breve\psi'_\pm(z,x)$.

(ii) *At the edges of the spectrum these functions possess the properties*

$$\psi_\pm(z,x) - \breve\psi_\pm(z,x) = O\left(\sqrt{z-E}\right) \quad \text{near} \quad E\in\partial\sigma_\pm\setminus\hat M_\pm,$$

and

$$\psi_\pm(z,x) + \breve\psi_\pm(z,x) = O(1) \quad \text{near} \quad E\in\hat M_\pm\ .$$

(iii) *When $z\to\infty$ the following asymptotic behavior holds*[2]*:*

$$\psi_\pm(z,x) = \mathrm{e}^{\pm\mathrm{i}\sqrt{z}x}\left(1+O(z^{-1/2})\right) \quad \text{and} \quad \breve\psi_\pm(z,x) = \mathrm{e}^{\mp\mathrm{i}\sqrt{z}x}\left(1+O(z^{-1/2})\right).$$

(iv) *The functions $\psi_\pm(\lambda,x)$ form a complete orthogonal system on the spectrum with respect to the weight*

$$d\rho_\pm(\lambda) = \frac{1}{2\pi\mathrm{i}}g_\pm(\lambda)d\lambda, \tag{2.14}$$

namely

$$\oint_{\sigma_\pm}\overline{\psi_\pm(\lambda,y)}\psi_\pm(\lambda,x)d\rho_\pm(\lambda) = \delta(x-y), \tag{2.15}$$

where $\delta(x)$ is the Dirac delta distribution. Here we have used the notation

$$\oint_{\sigma_\pm} f(\lambda)d\rho_\pm(\lambda) := \int_{\sigma_\pm^{\mathrm u}} f(\lambda)d\rho_\pm(\lambda) - \int_{\sigma_\pm^{\mathrm l}} f(\lambda)d\rho_\pm(\lambda). \tag{2.16}$$

[2]Here $\mathrm{Im}(\sqrt{z})>0$ as $z\in\mathbb{C}\setminus\mathbb{R}_+$.

3. The direct scattering problem

Consider the equation

$$\left(-\frac{d^2}{dx^2}+q(x)\right)y(x)=z\,y(x),\quad z\in\mathbb{C}, \tag{3.1}$$

with a potential $q(x)$, satisfying condition (1.3). This equation has two solutions $\phi_\pm(z,x)$, the Jost solutions, that are asymptotically close as $x\to\pm\infty$ to the Weyl solutions of the background equations (2.1) and can be represented as (see [13, 14, 15]):

$$\phi_\pm(z,x)=\psi_\pm(z,x)\pm\int_x^{\pm\infty}K_\pm(x,y)\psi_\pm(z,y)\,dy, \tag{3.2}$$

where $K_\pm(x,y)$ are real-valued, continuously differentiable with respect to both parameters, and satisfy the estimate

$$|K_\pm(x,y)|\leq\ \pm C_\pm(x)\int_{\frac{x+y}{2}}^{\pm\infty}|q(t)-p_\pm(t)|\,dt. \tag{3.3}$$

Here $C_\pm(x)$ are continuous positive functions, monotonically decreasing (and, therefore, bounded) as $x\to\pm\infty$ (see Appendix A). For $\lambda\in\sigma_\pm^{\mathrm{u}}\cup\sigma_\pm^{\mathrm{l}}$ a second pair of solutions of (3.1) is given by

$$\overline{\phi_\pm(\lambda,x)}=\breve\psi_\pm(\lambda,x)\pm\int_x^{\pm\infty}K_\pm(x,y)\breve\psi_\pm(\lambda,y)\,dy,\quad \lambda\in\sigma_\pm^{\mathrm{u}}\cup\sigma_\pm^{\mathrm{l}}. \tag{3.4}$$

Note $\breve\psi_\pm(\lambda,x)=\overline{\psi_\pm(\lambda,x)}$ for $\lambda\in\sigma_\pm$.

We see that, by formulas (3.2), (3.3), (3.4), and (2.11),

$$W\big(\phi_\pm(\lambda),\overline{\phi_\pm(\lambda)}\big)=\pm g_\pm(\lambda)^{-1}. \tag{3.5}$$

Unlike the Jost solutions, the solutions (3.4) exist only on the upper and lower cuts of the spectrum of the corresponding background, and cannot be continued to the complex plane.

The Jost solutions $\phi_\pm$ are holomorphic in the domains $\mathbb{C}\backslash(\sigma_\pm\cup M_\pm)$ and inherit almost all the properties of their background counterparts, listed in Lemma 2.1, (i)–(ii). In order to remove these singularities we introduce

$$\begin{aligned}\delta_\pm(z)&:=\prod_{\mu_j^\pm\in M_\pm}(z-\mu_j^\pm),\\ \hat\delta_\pm(z)&:=\prod_{\mu_j^\pm\in M_\pm}(z-\mu_j^\pm)\prod_{\mu_j^\pm\in\hat M_\pm}\sqrt{z-\mu_j^\pm},\\ \breve\delta_\pm(z)&:=\prod_{\mu_j^\pm\in\breve M_\pm}(z-\mu_j^\pm)\prod_{\mu_j^\pm\in\hat M_\pm}\sqrt{z-\mu_j^\pm},\end{aligned} \tag{3.6}$$

where $\prod=1$ if there are no multipliers, and set

$$\tilde\phi_\pm(z,x)=\delta_\pm(z)\phi_\pm(z,x),\quad \hat\phi_\pm(z,x)=\hat\delta_\pm(z)\phi_\pm(z,x). \tag{3.7}$$

Lemma 3.1. *The Jost solutions $\phi_\pm(z,x)$ have the following properties.*

(i) *For all x, the function $\phi_\pm(z,x)$ considered as function of z, is holomorphic in the domain $\mathbb{C}\setminus(\sigma_\pm\cup M_\pm)$, takes real values on the set $\mathbb{R}\setminus\sigma_\pm$, and has simple poles at the points of the set $M_\pm$. Moreover, $\hat\phi_\pm$ is continuous up to the boundary $\sigma_\pm^{\mathrm{u}}\cup\sigma_\pm^{\mathrm{l}}$.*

(ii) *$\phi_\pm(\lambda,x)-\overline{\phi_\pm(\lambda,x)}=O\left(\sqrt{\lambda-E}\right)$ for $E\in\partial\sigma_\pm\setminus\hat M_\pm$, and $\phi_\pm(\lambda,x)+\overline{\phi_\pm(\lambda,x)}=O(1)$ for $E\in\hat M_\pm$.*

Proof. Proof of this Lemma follows directly from (3.2), (3.3), (3.4), Lemma 2.1. □

Now introduce the sets

$$\sigma^{(2)}:=\sigma_+\cap\sigma_-,\quad \sigma_\pm^{(1)}=\mathrm{clos}(\sigma_\pm\setminus\sigma^{(2)}),\quad \sigma:=\sigma_+\cup\sigma_-, \tag{3.8}$$

where σ is the (absolutely) continuous spectrum of H and $\sigma_+^{(1)}\cup\sigma_-^{(1)}$, respectively $\sigma^{(2)}$ are the parts which are of multiplicity one, respectively two. In addition to the continuous part, H has a finite number of eigenvalues situated in the gaps, $\sigma_d\subset\mathbb{R}\setminus\sigma$ (see, e.g., [32]). We will use the notation $\mathrm{int}(\sigma_\pm)$ for the interior of the spectrum, that is, $\mathrm{int}(\sigma_\pm):=\sigma_\pm\setminus\partial\sigma_\pm$.

The following example illustrates the various possible locations of the spectra together with the Dirichlet eigenvalues.

Example 3.2. Let H_+ be the two-band quasi-periodic operator with the spectrum on the set $\sigma_+=[E_1,E_2]\cup[E_4,+\infty)$ and H_- be the three band operator with the spectrum $\sigma_-=[E_1,E_2]\cup[E_3,E_4]\cup[E_5,+\infty)$, where $E_1<E_2<\cdots<E_5$ (cf. Figure 1). Then $\sigma=[E_1,E_2]\cup[E_3,+\infty)$, $\sigma_+^{(1)}=[E_4,E_5]$, $\sigma_-^{(1)}=[E_3,E_4]$, and $\sigma^{(2)}=[E_1,E_2]\cup[E_5,+\infty)$. Denote by μ_1^- the Dirichlet eigenvalue for the operator H_-, that belongs to the closed gap $[E_2,E_3]$ and let μ_1^+ be the Dirichlet eigenvalue of H_+ from the gap $[E_2,E_4]$.

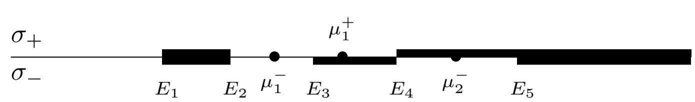

FIGURE 1. Typical mutual locations of σ_- and σ_+.

We may encounter various mutual locations of these eigenvalues. For example:

- $\mu_1^+\neq\mu_1^-$ and $\mu_1^+,\mu_1^-\in(E_2,E_3)$ (i.e., $\mu_1^\pm\in M_\pm$),
- $\mu_1^+=\mu_1^-\in(E_2,E_3)$,
- $\mu_1^+=E_2,\mu_1^-\neq E_2$ (i.e. $\mu_1^+\in\hat M_+$),
- $\mu_1^+\in[E_3,E_4]$ (the Dirichlet eigenvalue is situated on the spectrum of multiplicity one) and $\mu_1^-\neq E_3$,
- etc.

All such mutual locations of the Dirichlet eigenvalues imply different properties of the scattering data and have to be studied separately.

Let

$$W(z):=W(\phi_-(z,\cdot),\phi_+(z,\cdot)) \tag{3.9}$$

be the Wronskian of two Jost solutions. This function is meromorphic in the domain $\mathbb{C}\setminus\sigma$ with possible poles at the points $M_+\cup M_-\cup(\hat M_+\cap\hat M_-)$ and with possible square root singularities at the points $\hat M_+\cup\hat M_-\setminus(\hat M_+\cap\hat M_-)$. Set

$$\tilde W(z)=W(\tilde\phi_-(z),\tilde\phi_+(z)),\quad \hat W(z)=W(\hat\phi_-(z),\hat\phi_+(z)). \tag{3.10}$$

Then the function $\hat W(\lambda)$ is holomorphic in the domain $\mathbb C\setminus\mathbb R$ and continuous up to the boundary. But unlike the functions $W(z)$ and $\tilde W(z)$, it may not take real values on the set $\mathbb R\setminus\sigma$ and complex conjugated values on the different sides of the spectrum. That is why it is more convenient to characterize the spectral properties of the operator H by means of the function $\tilde W$, which can have singularities at the points of the set $\hat M_+\cup\hat M_-$. We will study the precise character of these singularities below.

Since the discrete spectrum of our operator H is finite, we can write it as

$$\sigma_d=\{\lambda_1,\dots,\lambda_p\}\subset\mathbb R\setminus\sigma.$$

For every eigenvalue we introduce the corresponding norming constants

$$\left(\gamma_k^\pm\right)^{-2}=\int_{\mathbb R}\tilde\phi_\pm^2(\lambda_k,x)\,dx. \tag{3.11}$$

Note that outside the spectrum $\tilde W(z)=0$ vanishes precisely at the eigenvalues. However, it might also vanish inside the spectrum at points in $\partial\sigma_-\cup\partial\sigma_+$ and we will call such points virtual levels of the operator H

$$\sigma_v:=\{E\in\sigma\mid \hat W(E)=0\}. \tag{3.12}$$

We will show $\sigma_v\subseteq\partial\sigma\cup(\partial\sigma_+^{(1)}\cap\partial\sigma_-^{(1)})$ in Lemma 3.3 below. All other points E of the set $\partial\sigma_+\cup\partial\sigma_-$ correspond to the generic case $\hat W(E)\neq 0$.

Now we begin our study of the properties of the scattering matrix. Introduce the scattering relations

$$T_\mp(\lambda)\phi_\pm(\lambda,x)=\overline{\phi_\mp(\lambda,x)}+R_\mp(\lambda)\phi_\mp(\lambda,x),\quad \lambda\in\sigma_\mp^{\mathrm{u,l}}, \tag{3.13}$$

where the transmission and reflection coefficients are defined as usual,

$$T_\pm(\lambda):=\frac{W(\overline{\phi_\pm(\lambda)},\phi_\pm(\lambda))}{W(\phi_\mp(\lambda),\phi_\pm(\lambda))},\qquad R_\pm(\lambda):=-\frac{W(\phi_\mp(\lambda),\overline{\phi_\pm(\lambda)})}{W(\phi_\mp(\lambda),\,\phi_\pm(\lambda))},\quad \lambda\in\sigma_\pm^{\mathrm{u,l}}. \tag{3.14}$$

Their characteristic properties will be given in the following lemma.

Lemma 3.3. *For the entries of the scattering matrix the following properties are valid:*

- **I.** **(a)** $T_\pm(\lambda^{\mathrm u})=\overline{T_\pm(\lambda^{\mathrm l})}$ *and* $R_\pm(\lambda^{\mathrm u})=\overline{R_\pm(\lambda^{\mathrm l})}$ *for* $\lambda\in\sigma_\pm$.
  - **(b)** $\dfrac{T_\pm(\lambda)}{\overline{T_\pm(\lambda)}}=R_\pm(\lambda)$ *for* $\lambda\in\sigma_\pm^{(1)}$.
  - **(c)** $1-|R_\pm(\lambda)|^2=\dfrac{g_\pm(\lambda)}{g_\mp(\lambda)}\,|T_\pm(\lambda)|^2$ *for* $\lambda\in\sigma^{(2)}$.
  - **(d)** $T_\pm(\lambda)=1+O(|\lambda|^{-1/2})$ *and* $R_\pm(\lambda)=O(|\lambda|^{-1/2})$ *for* $\lambda\to\infty$.
  - **(e)** $\overline{R_\pm(\lambda)}T_\pm(\lambda)+R_\mp(\lambda)\overline{T_\pm(\lambda)}=0$ *for* $\lambda\in\sigma^{(2)}$.
- **II.** *The functions* $T_\pm(\lambda)$ *can be extended analytically to the domain* $\mathbb C\setminus(\sigma\cup M_\pm\cup\check M_\pm)$ *and satisfy*

$$\frac{-1}{T_+(z)g_+(z)}=\frac{-1}{T_-(z)g_-(z)}=:W(z), \tag{3.15}$$

where $W(z)$ *possesses the following properties:*

(a) *The function* $\tilde W(z) = \delta_+(z)\delta_-(z)W(z)$ *is holomorphic in the domain* $\mathbb{C}\setminus\sigma$*, with simple zeros at the points* λ_k*, where*

$$\left(\frac{d\tilde W}{dz}(\lambda_k)\right)^2 = (\gamma_k^+\gamma_k^-)^{-2}. \tag{3.16}$$

Besides, it satisfies

$$\overline{\tilde W(\lambda^{\mathrm{u}})} = \tilde W(\lambda^{\mathrm{l}}), \quad \lambda\in\sigma \quad \textit{and} \quad \tilde W(\lambda)\in\mathbb{R} \quad \textit{for} \quad \lambda\in\mathbb{R}\setminus\sigma. \tag{3.17}$$

(b) *The function* $\hat W(z) = \hat\delta_+(z)\hat\delta_-(z)W(z)$ *is continuous on the set* $\mathbb{C}\setminus\sigma$ *up to the boundary* $\sigma^{\mathrm{u}}\cup\sigma^{\mathrm{l}}$*. It can have zeros on the set* $\partial\sigma\cup(\partial\sigma_+^{(1)}\cap\partial\sigma_-^{(1)})$ *and does not vanish at the other points of the set* σ*. If* $\hat W(E)=0$ *as* $E\in\partial\sigma\cup(\partial\sigma_+^{(1)}\cap\partial\sigma_-^{(1)})$*, then* $\hat W(z)=\sqrt{z-E}(C(E)+o(1))$*,* $C(E)\neq 0$*.*

III. (a) *The reflection coefficient* $R_\pm(\lambda)$*, is a continuous function on the set* $\mathrm{int}(\sigma_\pm^{\mathrm{u,l}})$*.*

(b) *If* $E\in\partial\sigma_+\cap\partial\sigma_-$ *and* $E\notin\sigma_v$ *then the function* $R_\pm(\lambda)$ *is also continuous at* E*. Moreover*

$$R_\pm(E) = \begin{cases} -1 & \textit{for } E\notin\hat M_\pm, \\ 1 & \textit{for } E\in\hat M_\pm. \end{cases} \tag{3.18}$$

Proof. The proof is based on formulas (3.14), (3.5), (3.2), and Lemma 3.1.

I. The symmetry property **(a)** follows from formulas (3.14), (2.13), (3.2), and (3.4). To verify **(b)** observe, that $\tilde\phi_\mp(\lambda,x)\in\mathbb{R}$ as $\lambda\in\mathrm{int}(\sigma_\pm^{(1)})$. Together with (3.14) and (3.7) this implies **(b)**. Now let $\lambda\in\mathrm{int}(\sigma^{(2)})$. Then by (3.13)

$$|T_\pm|^2 W(\phi_\mp,\overline{\phi_\mp}) = (|R_\pm|^2-1)W(\phi_\pm,\overline{\phi_\pm})$$

and property **(c)** follows from (3.5) and (2.14). To prove **(d)** we use (3.5), (2.7) and (3.2). Then property (iii) of Lemma 2.1 allows us to proceed as in the proof of [26, Lemma 3.5.1]to obtain the necessary asymptotics. The consistency condition **(e)** and the identity (3.15), considered on $\mathrm{int}(\sigma^{(2)})$, can be derived directly from the definition (3.14).

II. **(a)**. Except for (3.16) everything follows from the corresponding properties of $\phi_\pm(z,x)$ and it remains to show (3.16). If $\hat W(\lambda_0)=0$ for some $\lambda_0\in\mathbb{C}\setminus\sigma$, then

$$\tilde\phi_\pm(\lambda_0,x) = c_\pm\tilde\phi_\mp(\lambda_0,x) \tag{3.19}$$

for some constants $c_\pm$ (depending on λ_0) and satisfying $c_-\,c_+=1$. In particular, each zero of $\tilde W$ (or $\hat W$) outside the continuous spectrum, is a point of the discrete spectrum of H and vice versa.

Let $\gamma_\pm$ be the norming constants defined in (3.11) for some point of the discrete spectrum λ_0. Proceeding as in [26] one obtains

$$W\Big(\tilde\phi_\pm(\lambda_0,0),\frac{d}{d\lambda}\tilde\phi_\pm(\lambda_0,0)\Big) = \int_0^{\pm\infty}\tilde\phi_\pm^2(\lambda_0,x)\,dx. \tag{3.20}$$

Equalities (3.19) and (3.20) imply

$$\begin{aligned}\gamma_\pm^{-2} &= \mp c_\pm^2 \int_0^{\mp\infty} \tilde\phi_\mp^2(\lambda_0,x)dx \pm \int_0^{\pm\infty} \tilde\phi_\pm^2(\lambda_0,x)dx\\ &= \mp\, c_\pm^2 W\Big(\tilde\phi_\mp(\lambda_0,0), \frac{d}{d\lambda}\tilde\phi_\mp(\lambda_0,0)\Big) \pm W\Big(\tilde\phi_\pm(\lambda_0,0), \frac{d}{d\lambda}\tilde\phi_\pm(\lambda_0,0)\Big)\\ &= c_\pm \frac{d}{d\lambda} W(\tilde\phi_-(\lambda_0), \tilde\phi_+(\lambda_0))\end{aligned} \tag{3.21}$$

and, since $c_- c_+ = 1$, we obtain (3.16).

Item **(b)** will be shown in Lemma B.4.

III, **(a)** follows from the corresponding properties of $\phi_\pm(z,x)$ and from **II, (b)**. To show **III, (b)** we use that by (3.14) the reflection coefficient has the representation

$$R_\pm(\lambda) = -\frac{W(\overline{\phi_\pm(\lambda)}, \phi_\mp(\lambda))}{W\left(\phi_\pm(\lambda), \phi_\mp(\lambda)\right)} = \pm \frac{W(\overline{\phi_\pm(\lambda)}, \phi_\mp(\lambda))}{W(\lambda)} \tag{3.22}$$

and is continuous on both sides of the set $\mathrm{int}(\sigma_\pm) \setminus (M_\mp \cup \hat M_\mp)$. Moreover,

$$|R_\pm(\lambda)| = \left| \frac{W(\overline{\hat\phi_\pm(\lambda)}, \hat\phi_\mp(\lambda))}{\hat W(\lambda)} \right|,$$

where the denominator does not vanish on the set $\sigma_\pm \setminus \sigma_v$. Hence $R_\pm(\lambda)$ is continuous on this set since both the numerator and denominator are.

Next, let $E \in \partial\sigma_\pm \setminus \sigma_v$ (in particular $\hat W(E) \neq 0$). Then, if $E \notin \hat M_\pm$, we use (3.22) in the form

$$R_\pm(\lambda) = -1 \mp \frac{\hat\delta_\pm(\lambda) W(\phi_\pm(\lambda) - \overline{\phi_\pm(\lambda)}, \hat\phi_\mp(\lambda))}{\hat W(\lambda)}, \tag{3.23}$$

which shows $R_\pm(\lambda) \to -1$ since $\phi_\pm(\lambda) - \overline{\phi_\pm(\lambda)} \to 0$ by Lemma 3.1 (ii). This settles the first case in (3.18). Similarly, if $E \in \hat M_\pm$, we use (3.22) in the form

$$R_\pm(\lambda) = 1 \pm \frac{\hat\delta_\pm(\lambda) W(\phi_\pm(\lambda) + \overline{\phi_\pm(\lambda)}, \hat\phi_\mp(\lambda))}{\hat W(\lambda)}, \tag{3.24}$$

which shows $R_\pm(\lambda) \to 1$ since $\hat\delta_\pm(\lambda) \to 0$ and $\phi_\pm(\lambda) + \overline{\phi_\pm(\lambda)} = O(1)$ by Lemma 3.1 (ii). This settles the second case in (3.18) as well. □

We note that the behavior of $T_\pm(z)$ near the boundary points of the spectra can be read off from

$$T_\pm(z) = \frac{-1}{g_\pm(z) W(z)} = -\frac{\hat\delta_\mp(z)}{\check\delta_\pm(z)} \frac{2\sqrt{\prod_{j=0}^{2r_\pm}(z - E_j^\pm)}}{\hat W(z)}. \tag{3.25}$$

4. The Gel'fand-Levitan-Marchenko equation

The aim of this section is to derive the inverse scattering problem equation (the Gel'fand-Levitan-Marchenko equation) and to discuss some additional properties of the scattering data, that are consequences of this equation.

Lemma 4.1. *The inverse scattering problem (the GLM) equation has the form*

$$K_\pm(x,y) + F_\pm(x,y) \pm \int_x^{\pm\infty} K_\pm(x,t)F_\pm(t,y)dt = 0, \quad \pm y > \pm x \tag{4.1}$$

where

$$\begin{aligned} F_\pm(x,y) = &\oint_{\sigma_\pm} R_\pm(\lambda)\,\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda) + \\ &+ \int_{\sigma_\mp^{(1),\mathrm{u}}} |T_\mp(\lambda)|^2 \psi_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\mp(\lambda) \\ &+ \sum_{k=1}^{p} (\gamma_k^\pm)^2 \tilde\psi_\pm(\lambda_k,x)\tilde\psi_\pm(\lambda_k,y). \end{aligned} \tag{4.2}$$

Proof. Consider the function

$$\begin{aligned} G_\pm(z,x,y) &= T_\pm(z)\phi_\mp(z,x)\psi_\pm(z,y)g_\pm(z) - \breve\psi_\pm(z,x)\psi_\pm(z,y)g_\pm(z) \\ &:= G'_\pm(z,x,y) + G''_\pm(z,x,y), \quad \pm y > \pm x, \end{aligned} \tag{4.3}$$

where x, y are considered as parameters. As a function of z it is meromorphic in the domain $\mathbb{C}\setminus\sigma$ with simple poles at the points λ_k of the discrete spectrum. It is continuous up to the boundary $\sigma^{\mathrm{u}} \cup \sigma^{\mathrm{l}}$, except for the points of the edges of background spectra, where

$$G_\pm(z,x,y) = O\big((z-E)^{-1/2}\big) \quad \text{as} \quad E \in \partial\sigma_+ \cup \partial\sigma_-. \tag{4.4}$$

Since for $z \to \infty$ we have

$$\begin{aligned} &\phi_\mp(z,x) = \mathrm{e}^{\mp\mathrm{i}\sqrt{z}x}\big(1 + O(z^{-1/2})\big), \quad \breve\psi_\mp(z,x) = \mathrm{e}^{\mp\mathrm{i}\sqrt{z}x}\big(1 + O(z^{-1/2})\big), \\ &\psi_\pm(z,y) = \mathrm{e}^{\pm\mathrm{i}\sqrt{z}y}\big(1 + O(z^{-1/2})\big), \quad T_\pm(z) = 1 + O(z^{-1/2}) \end{aligned}$$

and $g_\pm(z) = \frac{-1}{2\mathrm{i}\sqrt{z}} + O(z^{-1})$ then

$$G_\pm(z,x,y) = \mathrm{e}^{\pm\mathrm{i}\sqrt{z}\,(y-x)}O(z^{-1}), \quad \pm y > \pm x. \tag{4.5}$$

Consider a closed contour Γ_ε consisting of a large circular arc together with some parts wrapping around the spectrum σ inside this arc at a small distance from the spectrum. By the Cauchy theorem

$$\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_\varepsilon} G_\pm(z,x,y)dz = \sum_{\lambda_k\in\sigma_d} \mathrm{Res}_{\lambda_k}\, G_\pm(z,x,y).$$

Estimate (4.5) allows us to apply Jordan's lemma. Since by (4.4) the limit value of $G_\pm$ as $\varepsilon \to 0$ is integrable on σ, and the function $G''_\pm$ has no poles at the points of the discrete spectrum, we arrive at

$$\frac{1}{2\pi\mathrm{i}} \oint_{\sigma} G_\pm(\lambda,x,y)d\lambda = \sum_{\lambda_k\in\sigma_d} \mathrm{Res}_{\lambda_k}\, G'_\pm(\lambda,x,y), \quad \pm y > \pm x. \tag{4.6}$$

Moreover, the function $G''_\pm$ also does not contribute to the left part of (4.6) since $G''_\pm(\lambda^{\mathrm{u}},x,y) = G''_\pm(\lambda^{\mathrm{l}},x,y)$ for $\lambda \in \sigma_\mp^{(1)}$ and, therefore, $\oint_{\sigma_\mp^{(1)}} G''_\pm(\lambda,x,y)d\lambda = 0$. In addition, $\oint_{\sigma_\pm} G''_\pm(\lambda,x,y)d\lambda = 0$ for $x \neq y$ by (2.15).

Next we study the contribution of the function $G'_\pm$. Properties **I, (b)** and **(c)** of Lemma 3.3 imply that

$$|R_\pm(\lambda)|<1 \quad \text{for} \quad \lambda\in\operatorname{int}(\sigma^{(2)}), \qquad |R_\pm(\lambda)|=1 \quad \text{for} \quad \lambda\in\sigma_\pm^{(1)}. \tag{4.7}$$

Now we consecutively use (3.15), (3.13), (3.14), (2.15), (3.2), (3.4) and again (2.15), to obtain

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}}\oint_{\sigma_\pm} G'_\pm(\lambda,x,y)d\lambda &= \oint_{\sigma_\pm} T_\pm(\lambda)\phi_\mp(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda)\\
&= \oint_{\sigma_\pm}\Big(R_\pm(\lambda)\phi_\pm(\lambda,x)+\overline{\phi_\pm(\lambda,x)}\Big)\psi_\pm(\lambda,y)d\rho_\pm(\lambda)\\
&= \oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda)+\oint_{\sigma_\pm}\breve{\psi}_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda)\\
&\quad \pm\int_x^{\pm\infty} dt\, K_\pm(x,t)\left(\oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,t)\psi_\pm(\lambda,y)d\rho_\pm(\lambda)+\delta(t-y)\right)\\
&= F_{r,\pm}(x,y)\pm\int_x^{\pm\infty} K_\pm(x,t)F_{r,\pm}(t,y)dt+K_\pm(x,y),
\end{aligned} \tag{4.8}$$

where

$$F_{r,\pm}(x,y)=\oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda). \tag{4.9}$$

On the set $\sigma_\mp^{(1)}$ both the numerator and denominator of the function $G'_\pm$ have poles (resp., square root singularities) at points of the set $\sigma_\mp^{(1)}\cap(M_\pm\cup(\partial\sigma_+^{(1)}\cap\partial\sigma_-^{(1)}))$ (resp. $\sigma_\mp^{(1)}\cap(M_\mp\setminus(M_\mp\cap M_\pm))$), but multiplying them, if necessary, by the functions (3.6), we can avoid singularities. Hence, without loss of generality, we can suppose $\sigma_\mp^{(1)}\cap(M_{r_+}\cup M_{r_-})=\emptyset$. Then, since $\psi_\pm(\lambda,x)\in\mathbb{R}$ as $\lambda\in\sigma_\mp^{(1)}$,

$$\frac{1}{2\pi\mathrm{i}}\oint_{\sigma_\mp^{(1)}} G'_\pm(\lambda,x,y)d\lambda=\frac{1}{2\pi\mathrm{i}}\int_{\sigma_\mp^{(1),\mathrm{u}}}\psi_\pm(\lambda,y)\left(\frac{\overline{\phi_\mp(\lambda,x)}}{\overline{W(\lambda)}}-\frac{\phi_\mp(\lambda,x)}{W(\lambda)}\right)d\lambda. \tag{4.10}$$

Property **I, (b)** of Lemma 3.3 and (3.13) imply

$$\overline{\phi_\mp(\lambda,x)}=T_\mp(\lambda)\phi_\pm(\lambda,x)-\frac{T_\mp(\lambda)}{\overline{T_\mp(\lambda)}}\phi_\mp(\lambda,x).$$

Therefore,

$$\begin{aligned}
\frac{\phi_\mp(\lambda,x)}{W(\lambda)}-\frac{\overline{\phi_\mp(\lambda,x)}}{\overline{W(\lambda)}} &= \phi_\mp(\lambda,x)\left(\frac{1}{W(\lambda)}+\frac{T_\mp(\lambda)}{\overline{T_\mp(\lambda)}\,\overline{W(\lambda)}}\right)-\frac{T_\mp(\lambda)\phi_\pm(\lambda,x)}{\overline{W(\lambda)}}\\
&= \phi_\mp(\lambda,x)\frac{2\operatorname{Re}\Big(T_\mp^{-1}(\lambda)\overline{W(\lambda)}\Big)T_\mp(\lambda)}{|W(\lambda)|^2}-\frac{T_\mp(\lambda)\phi_\pm(\lambda,x)}{\overline{W(\lambda)}}.
\end{aligned} \tag{4.11}$$

But by (3.15)

$$T_\mp^{-1}(\lambda)\overline{W(\lambda)}=|W(\lambda)|^2 g_\mp(\lambda)\in\mathrm{i}\mathbb{R}, \quad \text{for} \quad \lambda\in\sigma_\mp^{(1)},$$

thus, the first summand in (4.11) vanishes. And using $\overline{W}=(\overline{T_\mp}g_\mp)^{-1}$ we arrive at

$$\frac{\overline{\phi_\mp(\lambda,x)}}{\overline{W(\lambda)}}-\frac{\phi_\mp(\lambda,x)}{W(\lambda)}=|T_\mp(\lambda)|^2 g_\mp(\lambda)\phi_\pm(\lambda,x). \tag{4.12}$$

Combining (4.12), (2.14), (4.10), (3.2) and (4.8) we have

$$\frac{1}{2\pi \mathrm{i}}\oint_\sigma G_\pm(\lambda,x,y)d\lambda = F_{c,\pm}(x,y)+K_\pm(x,y)\pm\int_x^{\pm\infty}K_\pm(x,t)F_{c,\pm}(t,y)dt, \tag{4.13}$$

where

$$F_{c,\pm}(x,y):=F_{r,\pm}(x,y)+F_{h,\pm}(x,y), \tag{4.14}$$

$$F_{h,\pm}(x,y):=\int_{\sigma_\mp^{(1),\mathrm{u}}}\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)|T_\mp(\lambda)|^2d\rho_\mp(\lambda). \tag{4.15}$$

To derive the part of the GLM equation kernel, that correspond to the discrete spectrum (for function $F_{c,\pm}$ index c means the part, corresponding to the continuous spectrum), we apply (3.7), (3.10), (3.19), (3.21) and (3.2) to the right hand side of (4.6). Then,

$$\begin{aligned}
\sum_{\lambda_k\in\sigma_d}\mathrm{Res}_{\lambda_k}\,G'_\pm(\lambda,x,y) &= -\sum_{\lambda_k\in\sigma_d}\mathrm{Res}_{\lambda_k}\frac{\tilde\phi_\mp(\lambda,x)\tilde\psi_\pm(\lambda,y)}{\tilde W(\lambda)}\\
&= -\sum_{\lambda_k\in\sigma_d}\frac{\tilde\phi_\pm(\lambda_k,x)\tilde\psi_\pm(\lambda_k,y)}{\tilde W'(\lambda_k)c_{\pm,k}} = -\sum_{\lambda_k\in\sigma_d}(\gamma_k^\pm)^2\tilde\phi_\pm(\lambda_k,x)\tilde\psi_\pm(\lambda_k,y)\\
&= -F_{d,\pm}(x,y)\mp\int_x^{\pm\infty}K_\pm(x,t)F_{d,\pm}(t,y)dt,
\end{aligned} \tag{4.16}$$

where

$$F_{d,\pm}(x,y):=\sum_{\lambda_k\in\sigma_d}(\gamma_k^\pm)^2\tilde\psi_\pm(\lambda_k,x)\tilde\psi_\pm(\lambda_k,y). \tag{4.17}$$

Combining (4.6), (4.13), and (4.16) we finally obtain (4.2). □

As is shown in Appendix A, the kernel $F_\pm(x,y)$ of the GLM equation satisfies

Lemma 4.2. *The kernel of the GLM equation possess the following properties:*

IV. *The function $F_\pm(x,y)$ is continuously differentiable with respect to both variables and there exists real-valued function $q_\pm(x)$, $x\in\mathbb{R}$, with*

$$\pm\int_a^{\pm\infty}(1+x^2)|q_\pm(x)|dx<\infty, \quad \textit{for all } a\in\mathbb{R},$$

such that

$$|F_\pm(x,y)|\le C_\pm(x)Q_\pm(x+y), \tag{4.18}$$

$$\left|\frac{\partial}{\partial x}F_\pm(x,y)\right|\le C_\pm(x)\left(\left|q_\pm\left(\frac{x+y}{2}\right)\right|+Q_\pm(x+y)\right), \tag{4.19}$$

$$\pm\int_a^{\pm\infty}\left|\frac{d}{dx}F_\pm(x,x)\right|(1+x^2)\,dx<\infty, \tag{4.20}$$

where

$$Q_\pm(x):=\pm\int_{\frac{x}{2}}^{\pm\infty}|q_\pm(t)|\,dt, \tag{4.21}$$

and $C_\pm(x)>0$ is a continuous function, which decreases monotonically as $x\to\pm\infty$.

In summary, we have obtained the following necessary conditions for the scattering data:

Theorem 4.3 (necessary conditions for the scattering data)**.** *The scattering data*

$$\begin{aligned} \mathcal{S} = \Big\{ & R_+(\lambda),\, T_+(\lambda),\, \lambda\in\sigma_+^{\mathrm{u,l}};\, R_-(\lambda),\, T_-(\lambda),\, \lambda\in\sigma_-^{\mathrm{u,l}}; \\ & \lambda_1,\dots,\lambda_p\in\mathbb{R}\setminus(\sigma_+\cup\sigma_-),\, \gamma_1^\pm,\dots,\gamma_p^\pm\in\mathbb{R}_+\Big\} \end{aligned} \tag{4.22}$$

possess the properties **I-III** *listed in Lemma 3.3. The functions* $F_\pm(x,y)$*, defined in* (4.2)*, possess property* **IV** *from Lemma 4.2.*

In fact, the conditions on the scattering data, given in this theorem are both necessary and sufficient for the solution of the scattering problem in the class (1.1)–(1.3). The sufficiency of these conditions together with the algorithm for the solution of the inverse problem will be discussed in the next section.

As a consequence of the GLM equation and its unique solvability (see Lemma 5.1) and also formula (A.19) we note

Corollary 4.4. *Suppose* $q(x)$ *satisfies* (1.3)*, then* $q(x)$ *is uniquely determined by one of the sets of its "partial" scattering data* $\mathcal{S}_+$ *or* $\mathcal{S}_-$*, where*

$$\begin{aligned} \mathcal{S}_\pm = \Big\{ & R_\pm(\lambda),\, \lambda\in\sigma_\pm^{\mathrm{u}};\, |T_\mp(\lambda)|^2,\, \lambda\in\sigma_\mp^{(1),\mathrm{u}}; \\ & \lambda_1,\dots,\lambda_p\in\mathbb{R}\setminus(\sigma_+\cup\sigma_-),\, \gamma_1^\pm,\dots,\gamma_p^\pm\in\mathbb{R}_+\Big\}. \end{aligned} \tag{4.23}$$

The question about the characterization of such sets (necessary and sufficient conditions) for potentials from the class (1.3) is rather complicated and is still open.

5. The inverse scattering problem

Let $H_\pm$ be two one-dimensional finite-gap Schrödinger operators associated with potentials $p_\pm(x)$ as introduced in Section 2. Let $\mathcal{S}$ be given scattering data with corresponding kernels $F_\pm(x,y)$ satisfying the necessary conditions from Theorem 4.3.

We begin by showing that, given $F_\pm(x,y)$, the GLM equations (4.1) can be solved for $K_\pm(x,y)$.

Lemma 5.1. *Under condition* **IV**, (4.18)*, the GLM equations* (4.1) *have unique real-valued solutions* $K_\pm(x,\cdot\,)\in L^1(x,\pm\infty)$ *satisfying the estimates*

$$|K_\pm(x,y)|\le C_\pm(x)Q_\pm(x+y),\quad \pm y>\pm x. \tag{5.1}$$

Here the functions $Q_\pm(x)$ *are the same, as in* (4.21)*, and* $C_\pm(x)$ *are functions of the same type, as in Lemma 4.2 (i.e. positive, continuous and decreasing as* $x\to\pm\infty$*).*

Moreover, under condition **IV**, (4.19)*, the functions* $K_\pm(x,y)$ *are differentiable with respect to each variable and satisfy the estimate* (A.22)*, where the functions* $q_\pm(x)$ *are the same as in* (4.19) *and the functions* $C_\pm(x)$ *are of the same type as in* (5.1)*. Besides,*

$$\pm\int_a^{\pm\infty}(1+x^2)\left|\frac{d}{dx}K_\pm(x,x)\right|dx<\infty,\qquad \forall a\in\mathbb{R}. \tag{5.2}$$

Proof. The solvability of (4.1) under condition (4.18) together with the estimate (5.1) follows from considerations completely analogous to those ones used in the proof of Lemma A.3 (see Remark A.4). To prove uniqueness, first note that the GLM equations are generated by compact operators. Thus it is sufficient to prove, that the equation

$$f(x) \pm \int_x^{\pm\infty} F_\pm(x,y) f(y) dy = 0 \tag{5.3}$$

has only the trivial solution in the space $L^1(x, \pm\infty)$. The proof is similar for the "+" and "−" cases, hence we give it only for the "+" case. Let $f(y)$, $y > x$, be a nontrivial solution of (5.3) and set $f(y) = 0$ for $y \le x$. Since $F_+(x,y)$ is real-valued, we can assume $f(y)$ is real-valued. Abbreviate by

$$\widehat{f}(\lambda) = \int_{\mathbb{R}} \psi_+(\lambda, y) f(y) dy \tag{5.4}$$

the generalized Fourier transform, generated by the spectral decomposition (2.15) (cf. [34]). Recall that $\widehat{f}(\lambda) \in L^1_{\mathrm{loc}}(\sigma_+^{\mathrm{u}} \cup \sigma_+^{\mathrm{l}})$ and $\widehat{f}(\lambda) = O(1)$ as $\lambda \to +\infty$.

Multiplying (5.3) by $f(x)$, integrating over $\mathbb{R}$, and applying (5.4) and (4.2) we have

$$\begin{aligned} &2\int_{\sigma_+^u} |\widehat{f}(\lambda)|^2 d\rho_+(\lambda) + 2\,\mathrm{Re}\int_{\sigma_+^u} R_+(\lambda)\widehat{f}(\lambda)^2 d\rho_+(\lambda) \\ &+ \int_{\sigma_-^{(1),u}} \widehat{f}(\lambda)^2 |T_-(\lambda)|^2 d\rho_-(\lambda) + \sum_{k=1}^p (\gamma_k^+)^2 \left(\int_{\mathbb{R}} \tilde{\psi}_+(\lambda_k, y) f(y) dy \right)^2 = 0. \end{aligned} \tag{5.5}$$

The last two summands in (5.5) are nonnegative since $\widehat{f}(\lambda) \in \mathbb{R}$ for $\lambda \in \sigma_-^{(1)}$ and $\tilde{\psi}_+(\lambda_k, x) \in \mathbb{R}$. Ignoring the last one and proceeding as in [26, Lemma 3.5.3] we obtain

$$2\int_{\sigma^{(2),\mathrm{u}}} (1 - |R_+(\lambda)|) |\widehat{f}(\lambda)|^2 d\rho_+(\lambda) + \int_{\sigma_-^{(1),\mathrm{u}}} \widehat{f}(\lambda)^2 |T_-(\lambda)|^2 d\rho_-(\lambda) \le 0. \tag{5.6}$$

Here we used that

$$\int_{\sigma_+^{(1),\mathrm{u}}} (1 - |R_+(\lambda)|) |\widehat{f}(\lambda)|^2 d\rho_+(\lambda) = 0$$

by condition **I, (b)**. Now, since $|R_+(\lambda)| < 1$, $\rho_+(\lambda) > 0$ for $\lambda \in \mathrm{int}(\sigma^{(2)})$ and $\rho_-(\lambda) > 0$ for $\lambda \in \mathrm{int}(\sigma_-^{(1)})$, we conclude that

$$\widehat{f}(\lambda) = 0 \quad \text{for} \quad \lambda \in \sigma^{(2)} \cup \sigma_-^{(1)} = \sigma_-.$$

The function $\widehat{f}(z)$ can be defined by formula (5.4) as a meromorphic function on $\mathbb{C} \setminus \sigma_+$. By our analysis it is even meromorphic on $\mathbb{C} \setminus \sigma_+^{(1)}$ and vanishes on σ_-, thus $\widehat{f}(z)$ is equal to zero and hence also $f(x)$.

The estimate (5.2) follows by literally repeating the proof of Lemma A.3. □

Next, define two functions

$$\tilde{q}_\pm(x) = \mp 2 \frac{d}{dx} K_\pm(x,x) + p_\pm(x), \quad x \in \mathbb{R} \tag{5.7}$$

and note that estimate (5.2) implies

$$\pm \int_a^{\pm\infty} |\tilde{q}_\pm(x) - p_\pm(x)|(1 + x^2)dx < \infty, \quad a \in \mathbb{R}. \tag{5.8}$$

Lemma 5.2. *The functions* $\phi_\pm(z,x)$*, defined by*

$$\phi_\pm(z,x) = \psi_\pm(z,x) \pm \int_x^{\pm\infty} K_\pm(x,y)\psi_\pm(z,y)\,dy, \tag{5.9}$$

solve the equations

$$\left(-\frac{d^2}{dx^2} + \tilde{q}_\pm(x)\right) \phi_\pm(z,x) = z\phi_\pm(z,x), \tag{5.10}$$

where $\tilde{q}_\pm(x)$ *are defined by* (5.7)*.*

Proof. Consider the two operators[3]

$$\tilde{H}_\pm = -\frac{d^2}{dx^2} + \tilde{q}_\pm(x), \quad x \in \mathbb{R}.$$

On the corresponding half-axes the potentials $\tilde{q}_\pm(x)$ are asymptotically close to our background potentials $p_\pm(x)$. Define the integral operators

$$(\mathcal{K}_\pm f)(x) = f(x) \pm \int_x^{\pm\infty} K_\pm(x,y)f(y)dy.$$

Under the assumption, that the kernel $F_\pm(x,y)$ of the GLM equation is twice continuously differentiable, we infer from (4.1) that the function $K_\pm(x,y)$ is also twice differentiable (see the proof of Lemma A.3). Moreover one can prove, literally following [14], that the identity

$$\tilde{H}_\pm \mathcal{K}_\pm = \mathcal{K}_\pm H_\pm,$$

is valid. This identity implies (5.10). To obtain equality (5.10) without assumption of existence of the second derivatives, one can literally follow the proof of [26, Theorem 3.3.1]. □

The remaining problem is to show $\tilde{q}_+(x) \equiv \tilde{q}_-(x)$ under conditions **II** and **III** on the scattering data $\mathcal{S}$.

Theorem 5.3 (uniqueness of the reconstructed potential)**.** *Let the scattering data* $\mathcal{S}$*, defined as in* (4.22)*, satisfy conditions* **I, (a)–(d)**, **II**, **III, (a)** *and* **IV***. Then each of the GLM equations* (4.1) *has a unique solution* $K_\pm(x,y)$*, satisfying the estimate* (5.2)*. The functions* $\tilde{q}_\pm(x)$*, defined by* (5.7)*, satisfy* (5.8)*.*

Under additional conditions **III, (b)** *and* **I, (e)***, these two functions coincide,* $\tilde{q}_-(x) \equiv \tilde{q}_+(x) =: q(x)$*, and the data* $\mathcal{S}$ *are the scattering data for the Schrödinger operator with potential* $q(x)$*.*

Proof. To prove the uniqueness of the reconstructed potential we follow the method proposed in [26]. Recall that, according to [25, 34], we have the inversion formula for

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[3]We don't know $\tilde{H}_\pm$ is limit point at $\mp\infty$ yet, but this will not be used.

the generalized Fourier transform, generated by the spectral decomposition (2.15) and applied to the function $f(\lambda)\in L^1_{\mathrm{loc}}(\sigma_\pm^{\mathrm{u}}\cup\sigma_\pm^{\mathrm{l}})$, $f(\lambda)=O(1)$, $\lambda\to+\infty$:

$$\check f(y)=\oint_{\sigma_\pm} f(\lambda)\psi_\pm(\lambda,y)d\rho_\pm(\lambda),$$

$$f(\lambda)=\int_{\mathbb{R}}\check f(y)\overline{\psi_\pm(\lambda,y)}dy. \tag{5.11}$$

Split the kernel of the GLM equation (4.2) according to $F_\pm(x,y)=F_{r,\pm}(x,y)+F_{h,\pm}(x,y)+F_{d,\pm}(x,y)$ (cf. (4.9), (4.15), (4.17)).

We begin by considering the following part of the GLM equation

$$G_\pm(x,y):=F_{r,\pm}(x,y)\pm\int_x^{\pm\infty}K_\pm(x,t)F_{r,\pm}(t,y)dt, \tag{5.12}$$

where $K_\pm(x,y)$ are the solutions of GLM equations. By condition **I, (b)–(c)** we have $|R_\pm(\lambda)|\le 1$ and properties (iii) and (i) of Lemma 2.1 imply, that we can take $f(\lambda)=R_\pm(\lambda)\psi_\pm(\lambda,x)$ in (5.11). Using (4.9) we obtain

$$\int_{\mathbb{R}}F_{r,\pm}(x,y)\overline{\psi_\pm(\lambda,y)}dy=R_\pm(\lambda)\psi_\pm(\lambda,x). \tag{5.13}$$

and (3.2) consequently implies

$$\int_{\mathbb{R}}G_\pm(x,y)\overline{\psi_\pm(\lambda,y)}dy=R_\pm(\lambda)\phi_\pm(\lambda,x),\quad \lambda\in\sigma_\pm^{\mathrm{u,l}}. \tag{5.14}$$

On the other hand, invoking the GLM equations we have for $\pm y>\pm x$,

$$\begin{aligned}G_\pm(x,y)=&-K_\pm(x,y)-F_{h,\pm}(x,y)-F_{d,\pm}(x,y)\\&\mp\int_{\sigma_\mp^{(1),\mathrm{u}}}d\rho_\mp(\lambda)|T_\mp(\lambda)|^2\psi_\pm(\lambda,y)\int_x^{\pm\infty}K_\pm(x,t)\psi_\pm(\lambda,t)dt\\&\mp\sum_{k=1}^p(\gamma_k^\pm)^2\tilde\psi_\pm(\lambda_k,y)\int_x^{\pm\infty}K_\pm(x,t)\tilde\psi_\pm(\lambda_k,t)dt.\end{aligned}$$

Again using (3.2) this gives

$$\begin{aligned}G_\pm(x,y)=&-K_\pm(x,y)-\int_{\sigma_\mp^{(1),\mathrm{u}}}|T_\mp(\lambda)|^2\psi_\pm(\lambda,y)\phi_\pm(\lambda,x)d\rho_\mp(\lambda)\\&-\sum_{k=1}^p(\gamma_k^\pm)^2\tilde\psi_\pm(\lambda_k,y)\tilde\phi_\pm(\lambda_k,x).\end{aligned} \tag{5.15}$$

Now we use this formula to evaluate

$$\begin{aligned}\int_{\mathbb{R}}G_\pm(x,y)\breve\psi_\pm(\lambda,y)dy=&\mp\int_x^{\mp\infty}G_\pm(x,y)\breve\psi_\pm(\lambda,y)dy\mp\int_x^{\pm\infty}K_\pm(x,y)\breve\psi_\pm(\lambda,y)dy\\&\pm\int_{\sigma_\mp^{(1),\mathrm{u}}}|T_\mp(\xi)|^2\phi_\pm(\xi,x)\,W(\psi_\pm(\xi,x),\breve\psi_\pm(\lambda,x))\frac{d\rho_\mp(\xi)}{\xi-\lambda}\\&\pm\sum_{k=1}^p(\gamma_k^\pm)^2\tilde\phi_\pm(\lambda_k,x)\,\frac{W(\tilde\psi_\pm(\lambda_k,x),\breve\psi_\pm(\lambda,x))}{\lambda_k-\lambda}.\end{aligned}$$

Here we have used

$$(\xi-\lambda)\int_x^{\pm\infty}\breve\psi_\pm(\lambda,y)\psi_\pm(\xi,y)dy=W(\breve\psi_\pm(\lambda,x),\psi_\pm(\xi,x)),$$

which follows from Green's formula and the fact, that $\tilde\psi_\pm(\xi,y)\to 0$, $\tilde\psi'_\pm(\xi,y)\to 0$ for $\xi\notin\sigma_\pm$ as $y\to\pm\infty$.

Combining this formula with (5.14) and using (3.4) we infer the relation

$$R_\pm(\lambda)\,\phi_\pm(\lambda,x)+\overline{\phi_\pm(\lambda,x)}=T_\pm(\lambda)\theta_\mp(\lambda,x),\quad \lambda\in\sigma_\pm^{\mathrm{u,l}}, \tag{5.16}$$

where

$$\begin{aligned}\theta_\mp(\lambda,x):=\frac{1}{T_\pm(\lambda)}\Bigg(&\breve\psi_\pm(\lambda,x)\mp\int_x^{\mp\infty}G_\pm(x,y)\breve\psi_\pm(\lambda,y)dy\\ &-\int_{\sigma_\mp^{(1),\mathrm{u}}}|T_\mp(\xi)|^2\phi_\pm(\xi,x)W(\psi_\pm(\xi,x),\breve\psi_\pm(\lambda,x))\frac{d\rho_\mp(\xi)}{\xi-\lambda}\\ &\pm\sum_{k=1}^p(\gamma_k^\pm)^2\tilde\phi_\pm(\lambda_k,x)\frac{W(\tilde\psi_\pm(\lambda_k,x),\breve\psi_\pm(\lambda,x))}{\lambda_k-\lambda}\Bigg).\end{aligned} \tag{5.17}$$

It turns out that, in spite of the fact that $\theta_\mp(\lambda,x)$ is defined via the background solutions corresponding to the opposite half-axis $\mathbb{R}_\pm$, it shares a series of properties with $\phi_\mp(\lambda,x)$. Namely, we prove

Lemma 5.4. *Let $\theta_\mp(z,x)$ be defined by formula (5.17) on the set $\sigma_\pm^{\mathrm{u,l}}$.*

(i) *The function $\tilde\theta_\mp(z,x)=\delta_\mp(z)\theta_\mp(z,x)$ admits an analytical extension to the domain $\mathbb{C}\setminus\sigma$.*

(ii) *The function $\tilde\theta_\mp(z,x)$ is continuous up to the boundary $\sigma^{\mathrm{u,l}}$ except possibly at the points $\partial\sigma_+\cup\partial\sigma_-$. Furthermore,*

$$\theta_\mp(\lambda^{\mathrm{u}},x)=\begin{cases}\theta_\mp(\lambda^{\mathrm{l}},x)\in\mathbb{R}, & \text{for } \lambda\in\mathbb{R}\setminus\sigma_\mp,\\ \overline{\theta_\mp(\lambda^{\mathrm{l}},x)}, & \text{for } \lambda\in\mathrm{int}(\sigma_\mp).\end{cases} \tag{5.18}$$

(iii) *For large z the function $\theta_\mp(z,x)$ has the following asymptotic behavior*

$$\theta_\mp(z,x)=\mathrm{e}^{\mp\mathrm{i}\sqrt{z}\,x}\big(1+O(z^{-1/2})\big),\quad z\to\infty.$$

(iv) $W\big(\theta_\mp(z,x),\phi_\pm(z,x)\big)=\pm W(z)$, *where $W(z)$ is defined by formula (3.15).*

Remark 5.5. Note that we did not establish the connection between the function $W(z)$ and the function $W\big(\phi_+(z,x),\phi_-(z,x)\big)$, which can depend on x, because ϕ_+ and ϕ_- are the solutions of Schrödinger equations corresponding to possibly different potentials $\tilde q_+$ and $\tilde q_-$.

Proof. To show (i) we will show that $\tilde\theta_\mp(z,x)$ has an analytic extension to $\mathbb{C}\setminus\sigma$. We will study each term in (5.17) separately.

For the first one,

$$\zeta_\mp(z,x):=\frac{\breve\psi_\pm(z,x)}{T_\pm(z)}, \tag{5.19}$$

it is immediate by (3.25) that $\tilde\zeta_\mp(z,x)=\delta_\mp(z)\zeta_\mp(z,x)$ has the required property. This also covers the second term since $G_\pm(x,\cdot\,)\in L^1(\mathbb{R})$ is real-valued.

Now we discuss the properties of the Cauchy-type integral in the representation (5.17). Multiplying it by $T_\pm^{-1}(z)$, we represent the third summand in (5.17) as

$$H_\mp(z,x):=\mp\frac{1}{2\pi\mathrm{i}}\int_{\sigma_\mp^{(1),\mathrm{u}}}h_\mp(z,\xi,x)\frac{d\xi}{\xi-z}, \tag{5.20}$$

where the integrand, due to (3.10), (2.14), and (3.15), has the representation

$$\begin{aligned} h_{\mp}(z,\xi,x) &= \frac{\delta_{\mp}(\xi)^2}{g_{\mp}(\xi)|\tilde{W}(\xi)|^2}\tilde{\phi}_{\pm}(\xi,x)W(\tilde{\psi}_{\pm}(\xi,x),\zeta_{\mp}(z,x)) \\ &= \frac{|\hat{\delta}_{\mp}(\xi)|^2}{g_{\mp}(\xi)|\hat{W}(\xi)|^2}\frac{|\hat{\delta}_{\pm}(\xi)|^2}{\hat{\delta}_{\pm}(\xi)^2}\hat{\phi}_{\pm}(\xi,x)W(\hat{\psi}_{\pm}(\xi,x),\zeta_{\mp}(z,x)). \end{aligned} \tag{5.21}$$

By property **II, (b)** the function $\hat{W}(\xi)$ has no zeros in the interior of $\sigma_{\mp}^{(1),\mathrm{u}}$. Thus, for $z \notin \sigma_{\mp}^{(1)}$, the function $h_{\mp}(z,.,x)$ is bounded in the interior of $\sigma_{\mp}^{(1)}$ and the only possible singularities can arise at the boundary. We claim

$$h_{\mp}(z,\xi,x) = \begin{cases} O(\sqrt{\xi-E}) & \text{for } E \notin \sigma_v, \\ O\left(\frac{1}{\sqrt{\xi-E}}\right) & \text{for } E \in \sigma_v, \end{cases} \qquad E \in \partial\sigma_{\mp}^{(1)},\ z \neq E. \tag{5.22}$$

This follows from $\frac{|\hat{\delta}_{\mp}(\xi)|^2}{g_{\mp}(\xi)} = O(\sqrt{\xi-E})$ together with $\hat{W}(\xi) = O(1)$ if $E \notin \sigma_v$ and $\hat{W}(\xi) = C(E)(\sqrt{\xi-E})(1+o(1))$, $C(E) \neq 0$ by **II, (b)** if $E \in \sigma_v$.

So $h_{\mp}$ is integrable and the third summand of (5.17) also inherits the properties of $\zeta_{\mp}(z,x)$.

Finally, let us consider the last summand in (5.17). It again inherits everything from $\tilde{\zeta}_{\mp}(z,x)$ except for possible additional poles at the points λ_k. However, these are canceled by the fact that the function $\tilde{W}(z)$ vanishes for $z = \lambda_k$.

(ii). Next we look at the boundary values. The only nontrivial term is of course the Cauchy-type integral (5.20) as $z \to \lambda \in \mathrm{int}(\sigma_{\mp}^{(1)})$. First of all observe that by (2.11) and (3.15) we have

$$\frac{W(\tilde{\psi}_{\pm}(\lambda,x),\breve{\psi}_{\pm}(z,x))}{T_{\pm}(z)} \to \pm(\delta_{\pm}W)(\lambda),$$

where the function $\delta_{\pm}W$ is bounded and non zero for $\lambda \in \mathrm{int}(\sigma_{\mp}^{(1)})$ by **II, (a)**. Hence the Plemelj formula applied to (5.20) gives

$$H_{\mp}(\lambda,x) = \pm\frac{\tilde{\phi}_{\pm}(\lambda,x)}{2\delta_{\pm}(\lambda)g_{\mp}(\lambda)\overline{W(\lambda)}} \mp \fint_{\sigma_{\mp}^{(1),\mathrm{u}}} \frac{h_{\mp}(\lambda,\xi,x)}{\xi-\lambda}d\xi, \quad \lambda \in \mathrm{int}(\sigma_{\mp}^{(1),\mathrm{u}}),$$

where both terms are finite. Here $\fint$ denotes the principal value integral.

Hence the boundary values away from $\partial\sigma_+ \cup \partial\sigma_-$ exist and we have

$$\theta_{\mp}(\lambda^{\mathrm{u}},x) = \overline{\theta_{\mp}(\lambda^{\mathrm{l}},x)}, \quad \lambda \in \sigma_+ \cup \sigma_-. \tag{5.23}$$

Moreover, by property **I, (b)** we have

$$\theta_{\mp} = T_{\pm}^{-1}\left(R_{\pm}\phi_{\pm} + \overline{\phi_{\pm}}\right) = \frac{\phi_{\pm}}{\overline{T_{\pm}}} + \frac{\overline{\phi_{\pm}}}{T_{\pm}} \in \mathbb{R} \quad \text{for} \quad \lambda \in \sigma_{\pm}^{(1)}, \tag{5.24}$$

from which

$$\theta_{\mp}(\lambda^{\mathrm{u}},x) = \theta_{\mp}(\lambda^{\mathrm{l}},x), \quad \lambda \in \sigma_{\pm}^{(1)}, \tag{5.25}$$

follows. Combining (5.23) and (5.25) we have (5.18).

(iii). For $|z| \to \infty$ due to properties (iii) of Lemma 2.1 and **I, (d)** we have

$$\zeta_{\mp}(z,x) = \mathrm{e}^{\mp\mathrm{i}\sqrt{z}x}\left(1 + O(z^{-1/2})\right). \tag{5.26}$$

Since the last two terms in (5.17) are $O(z^{-1})$ we obtain

$$\theta_\mp(z,x) = \mathrm{e}^{\mp \mathrm{i}\sqrt{z}x}\left(1 + \int_0^\infty G_\pm(x, x\mp t)\mathrm{e}^{\mathrm{i}\sqrt{z}t}dt + O(z^{-1/2})\right)$$

which implies (iii) since $G_\pm(x,y)$ is differentiable with respect to y.

(iv). From (5.16) (invoking (3.15)) we obtain

$$W\big(\theta_\mp(z,x), \phi_\pm(z,x)\big) = \pm W(z) \tag{5.27}$$

for $z\in\sigma_\pm$. Hence equality holds for all $z\in\mathbb{C}$ by analytical continuation. □

Corollary 5.6. *The function $\tilde\theta_\mp(z,x)$ admits an analytical extension to the domain $\mathbb{C}\setminus\sigma_\mp$.*

Proof. Property (i) holds for $z\in\mathbb{C}\setminus\sigma$. Relation (5.18) implies that $\tilde\theta_\mp$ has no jump across $z\in\operatorname{int}(\sigma_\pm^{(1)})$. To finish the proof we need to show that the possible remaining singularities at $E\in\partial\sigma_\pm^{(1)}\cap\partial\sigma$ are removable. This follows from (cf. (3.25))

$$\hat\zeta_\mp(z,x) = \frac{\hat W(z)}{2\sqrt{\prod_{j=0}^{2r_\pm}(z-E_j^\pm)}}\breve\delta_\pm(z)\breve\psi_\pm(z,x) \tag{5.28}$$

which shows $\tilde\zeta_\mp(z,x) = O((z-E)^{-1/2})$ and hence $\tilde\theta_\mp(z,x) = O((z-E)^{-1/2})$ for $E\in\sigma_\pm^{(1)}\cap\partial\sigma$.

However, let us emphasize at this point that the behavior of $\theta_\pm(z,x)$ at the remaining edges is a more subtle question to be discussed later. □

Eliminating $\overline{\phi_\pm}$ from

$$\begin{cases} \overline{R_\pm(\lambda)}\,\overline{\phi_\pm(\lambda,x)} + \phi_\pm(\lambda,x) = \overline{\theta_\mp(\lambda,x)}\,\overline{T_\pm(\lambda)} \\ R_\pm(\lambda)\,\phi_\pm(\lambda,x) + \overline{\phi_\pm(\lambda,x)} = \theta_\mp(\lambda,x)\,T_\pm(\lambda) \end{cases}$$

we obtain

$$\phi_\pm(\lambda,x)\left(1-|R_\pm(\lambda)|^2\right) = \overline{\theta_\mp(\lambda,x)}\,\overline{T_\pm(\lambda)} - \overline{R_\pm(\lambda)}\,\theta_\mp(\lambda,x)\,T_\pm(\lambda).$$

Next, using **I, (c)**, **II** and the consistency condition **I, (e)** then shows

$$\begin{aligned} T_\mp(\lambda)\phi_\pm(\lambda,x) &= \overline{\theta_\mp(\lambda,x)} - \frac{\overline{R_\pm(\lambda)}T_\pm(\lambda)}{\overline{T_\pm(\lambda)}}\theta_\mp(\lambda,x) \\ &= \overline{\theta_\mp(\lambda,x)} + R_\mp(\lambda)\theta_\mp(\lambda,x), \quad \lambda\in\sigma^{(2)}. \end{aligned}$$

This equation together with (5.16) gives us a system from which we can eliminate the reflection coefficients $R_\pm$. We obtain

$$T_\pm(\lambda)(\phi_\pm(\lambda)\phi_\mp(\lambda) - \theta_\pm(\lambda)\theta_\mp(\lambda)) = \phi_\pm(\lambda)\overline{\theta_\pm(\lambda)} - \overline{\phi_\pm(\lambda)}\theta_\pm(\lambda), \quad \lambda\in\sigma^{(2),\mathrm{u,l}}. \tag{5.29}$$

Now introduce the function

$$G(z) := G(z,x) = \frac{\phi_+(z,x)\phi_-(z,x) - \theta_+(z,x)\theta_-(z,x)}{W(z)} \tag{5.30}$$

which is well defined in the domain $z\in\mathbb{C}\setminus(\sigma\cup\sigma_d\cup M_+\cup M_-)$.

From (5.29) and (3.15) we see, that

$$G(\lambda) = -\left(\phi_\pm(\lambda)\overline{\theta_\pm(\lambda)} - \overline{\phi_\pm(\lambda)}\theta_\pm(\lambda)\right)g_\pm(\lambda), \qquad \lambda\in\sigma^{(2),\mathrm{u,l}}. \tag{5.31}$$

So we need to study the properties of $G(z,x)$ as a function of z (regarding x as a fixed parameter). Our aim is to prove that $G(z,x)=0$. This will follow from

Lemma 5.7. *The function $G(z,x)$ possess the following properties.*

(i)

$$G(\lambda^{\mathrm{u}},x)=G(\lambda^{\mathrm{l}},x)\in\mathbb{R} \text{ for } \lambda\in\mathbb{R}\setminus(\partial\sigma_-\cup\partial\sigma_+\cup\sigma_d). \tag{5.32}$$

(ii) *It has removable singularities at the points $\partial\sigma_-\cup\partial\sigma_+\cup\sigma_d$, where $\sigma_d:=\{\lambda_1,\dots,\lambda_p\}$.*

Proof. (i). We can rewrite $G(z,x)$ as

$$G(z,x)=\frac{\tilde\phi_+(z,x)\tilde\phi_-(z,x)-\tilde\theta_+(z,x)\tilde\theta_-(z,x)}{\tilde W(z)}, \tag{5.33}$$

where $\tilde\theta_\pm(z,x)=\delta_\pm(z)\theta_\pm(z,x)$ as usual. The numerator is bounded near the points under consideration, and the denominator does not vanish there. Thus $G(z,x)$ has no singularities at the points $(M_+\cup M_-)\setminus\sigma_d$.

Furthermore, by Lemma 5.4, **II, (a)**, and Lemma 3.1, (i) we know that $G(z,x)$ has continuous limiting values on the sets σ_- and σ_+, except possibly at the edges, satisfying

$$G(\lambda^{\mathrm{u}},x)=\overline{G(\lambda^{\mathrm{l}},x)},\quad \lambda\in\sigma_+\cup\sigma_-.$$

Hence, if we can show that these limits are real, they will be equal and $G(z,x)$ will extend to a meromorphic function on $\mathbb{C}$, that is, (i) holds.

First of all observe that from (5.18), (5.31), (2.7), and Lemma 3.1 (i), it follows, that

$$G(\lambda^{\mathrm{u}},x)=G(\lambda^{\mathrm{l}},x)\in\mathbb{R},\quad \lambda\in\operatorname{int}(\sigma^{(2)}). \tag{5.34}$$

Thus, it remains to prove

$$G(\lambda^{\mathrm{u}},x)=G(\lambda^{\mathrm{l}},x)\in\mathbb{R}\quad\text{for}\quad \lambda\in\operatorname{int}(\sigma_-^{(1)})\cup\operatorname{int}(\sigma_+^{(1)}). \tag{5.35}$$

So let us show that there is no jump on the set $\operatorname{int}(\sigma_-^{(1)})\cup\operatorname{int}(\sigma_+^{(1)})$. Abbreviate

$$[G]:=G(\lambda)-\overline{G(\lambda)}=\left[\frac{\phi_+\phi_-}{W}\right]-\left[\frac{\theta_+\theta_-}{W}\right],\qquad \lambda\in\sigma_\pm^{(1),\mathrm{u}}, \tag{5.36}$$

and let us drop some dependencies until the end of this lemma for notational simplicity.

Let $\lambda\in\operatorname{int}(\sigma_\mp^{(1),\mathrm{u}})$, then $\phi_\pm,\theta_\pm\in\mathbb{R}$ and $\overline{T}_\mp=(\overline{W}\,g_\mp)^{-1}$. Thus, by condition **(I), (b)** and (5.16) for $\lambda\in\operatorname{int}(\sigma_\mp^{(1)})$ we obtain

$$\left[\frac{\phi_+\phi_-}{W}\right]=\phi_\pm\left[\frac{\phi_\mp}{W}\right]=-g_\mp\phi_\pm\left(\phi_\mp T_\mp+\overline{\phi}_\mp\overline{T}_\mp\right)=-g_\mp\theta_\pm\phi_\pm|T_\mp|^2. \tag{5.37}$$

Since $g_\pm\in\mathbb{R}$ for $\lambda\in\operatorname{int}(\sigma_\mp^{(1),\mathrm{u}})$, (3.15) implies

$$\left[\frac{\theta_\mp}{W}\right]=-g_\pm\left[\theta_\mp T_\pm\right].$$

The only non-real summand in (5.17) is the Cauchy-type integral. The Plemelj formula applied to this integral gives

$$[\theta_\mp T_\pm]=\pm g_\mp\phi_\pm|T_\mp|^2W(\psi_\pm,\breve\psi_\pm)=g_\mp\phi_\pm|T_\mp|^2\frac{1}{g_\pm}.$$

Thus by (5.37)

$$\left[\frac{\theta_+\theta_-}{W}\right] = \left[\frac{\phi_+\phi_-}{W}\right] = -g_\mp\phi_\pm\theta_\pm|T_\mp|^2, \quad \lambda\in\operatorname{int}(\sigma_\mp^{(1)}). \tag{5.38}$$

Since $\tilde W\neq 0$ and $s_\mp\neq 0$ for $\lambda\in\operatorname{int}(\sigma_\mp^{(1)})$, the function

$$g_\mp\phi_\pm\theta_\pm|T_\mp|^2 = -\frac{\delta_\mp^2}{g_\mp}\frac{\tilde\phi_\pm\tilde\theta_\pm}{|\tilde W|^2}$$

is bounded on the set under consideration. Finally, (5.38) and (5.36) imply (5.35).

(ii) Now we prove, that the function $G(z,x)$ has removable singularities at the points $\partial\sigma_-\cup\partial\sigma_+\cup\sigma_d$. Divide this set in four subsets

$$\Omega_1^\pm = \partial\sigma^{(2)}\cap\operatorname{int}(\sigma_\mp),\ \Omega_2^\pm = \partial\sigma_\pm^{(1)}\cap\partial\sigma,\ \Omega_3=\partial\sigma_-\cap\partial\sigma_+,\ \text{and}\ \Omega_4=\sigma_d. \tag{5.39}$$

In our example we have $\Omega_1^+=\emptyset$, $\Omega_1^-=\{E_5\}$, $\Omega_2^+=\emptyset$, $\Omega_2^-=\{E_3\}$, and $\Omega_3=\{E_1,E_2,E_4\}$.

By condition **II, (b)** all singularities of G are at most isolated poles. Thus it is sufficient to check that

$$G(z) = o\left((z-E)^{-1}\right) \tag{5.40}$$

from some direction in the complex plane.

• Ω_1: Consider $E\in\Omega_1^+$ (the case $E\in\Omega_1^-$ is completely analogous). We will study $\lim_{\lambda\to E}G(\lambda,x)$ as $\lambda\in\operatorname{int}(\sigma^{(2)})$ using identity (5.31). We have $\phi_-=O(1)$, $g_-=O(1)$, and $\hat W(E)\neq 0$. Moreover, from Lemma 3.1 respectively **II** we deduce

$$\phi_+(\lambda) = \begin{cases} O(1), & E\notin\hat M_+,\\ O\left(\frac{1}{\sqrt{\lambda-E}}\right), & E\in\hat M_+,\end{cases} \qquad \frac{1}{T_+(\lambda)} = \begin{cases} O\left(\frac{1}{\sqrt{\lambda-E}}\right), & E\notin\hat M_+,\\ O(1), & E\in\hat M_+,\end{cases}$$

which shows

$$\theta_-(\lambda) = \frac{\overline{\phi_+(\lambda)}+R_+(\lambda)\phi_+(\lambda)}{T_+(\lambda)} = O\left(\frac{1}{\sqrt{\lambda-E}}\right),$$

for $\lambda\in\sigma^{(2)}$. Inserting this into (5.31) shows $G(\lambda,x)=O\left(\frac{1}{\sqrt{\lambda-E}}\right)$ and finishes the case $E\in\Omega_1$.

• Ω_2: Suppose now that $E\in\partial\sigma_-^{(1)}\cap\partial\sigma$ (the case $E\in\partial\sigma_+^{(1)}\cap\partial\sigma$ can be treated in the same manner). Now we cannot use (5.31), so we proceed directly from formula (5.30) estimating the summands $\frac{\phi_+\phi_-}{W}$ and $\frac{\theta_+\theta_-}{W}$ separately. By Lemma 3.1 and **II, (b)** we have

$$\frac{\phi_+\phi_-}{W} = \frac{\hat\phi_+\hat\phi_-}{\hat W} = O\left(\frac{1}{\sqrt{z-E}}\right). \tag{5.41}$$

Hence the first term is under control and it remains to investigate the second one. We investigate the limit of G from the set $\operatorname{int}(\sigma_-^{(1)})$. By (5.30)

$$\frac{\theta_+}{W} = \theta_+T_-g_- = (\overline{\phi_-}+\phi_-R_-)g_- = \begin{cases} O(1), & E\in\hat M_-,\\ O\left(\frac{1}{\sqrt{\lambda-E}}\right), & E\notin\hat M_-,\end{cases} \quad \lambda\in\operatorname{int}(\sigma_-). \tag{5.42}$$

Next we will estimate θ_- using (5.17). First, let $E\notin\sigma_v$, that is $\hat W(E)\neq 0$. Using the same notation, see (5.20), as in the proof of Lemma 5.4 we can split $\theta_-(\lambda)$

according to

$$\theta_-(\lambda) = \theta_1(\lambda) + \theta_2(\lambda), \tag{5.43}$$

where

$$\theta_2(\lambda) = \frac{1}{2\pi\mathrm{i}} \int_{\sigma_-^{(1),u}} h_-(\lambda,\xi) \frac{d\xi}{\xi-\lambda}, \qquad \theta_1(\lambda) = \theta_-(\lambda) - \theta_2(\lambda). \tag{5.44}$$

We see that (cf. (5.28))

$$\theta_1 = O(\zeta_-) = \frac{\hat{W}}{\hat{\delta}_-} O(1) = \begin{cases} O\left(\frac{1}{\sqrt{\lambda-E}}\right), & E \in \hat{M}_-, \\ O(1), & E \notin \hat{M}_-, \end{cases}$$

since $E \notin \sigma_+$. Next, we use (5.44), where (cf. (5.21))

$$h_-(\lambda,\xi) = \frac{\sqrt{\xi-E}}{|\hat{W}(\xi)|^2} C(\xi) O(\zeta_-(\lambda))$$

with $C(\xi)$ some bounded function near E. Hence $\theta_2 = O(\zeta_-)$ as well, which implies together with (5.42) that

$$\frac{\theta_+(\lambda)\theta_-(\lambda)}{W(\lambda)} = O\left(\frac{1}{\sqrt{\lambda-E}}\right).$$

This finishes the case $E \notin \sigma_v$.

Now let $E \in \sigma_v$. As before we have

$$\theta_1 = O(\zeta_-) = \begin{cases} O(1), & E \in \hat{M}_-, \\ O\left(\sqrt{\lambda-E}\right), & E \notin \hat{M}_-. \end{cases} \tag{5.45}$$

For the Cauchy-type integral we now have

$$h_-(\lambda,\xi) = \frac{C(\xi)}{\sqrt{\xi-E}} O(\zeta_-(\lambda))$$

and [27, Eq. (29.8)] implies

$$\theta_2(\lambda) = \begin{cases} o\left(\frac{1}{\sqrt{\lambda-E}}\right), & E \in \hat{M}_-, \\ o(1), & E \notin \hat{M}_-. \end{cases} \tag{5.46}$$

Combining (5.45), (5.46), and (5.42) finishes the second case.

• Ω_3: The first step is similar as in the case of Ω_2. In particular, (5.41) is valid for $E \in \Omega_3$ and $z \in \mathbb{C}$ close to E. Estimate (5.42) is valid for $\lambda \in \operatorname{int}(\sigma_-)$. The only difference being that ζ_- in estimate for θ_- has an additional square root singularity since $E \in \partial\sigma_+$. That is, instead of (5.45) and (5.46) we now have

$$\theta_-(z) = \zeta_-(z)(C + o(1)) = \begin{cases} O\left(\frac{1}{z-E}\right), & E \in \hat{M}_-, \\ O\left(\frac{1}{\sqrt{z-E}}\right), & E \notin \hat{M}_-, \end{cases} \qquad z \in \mathbb{C}.$$

However, this is not good enough unless we can show $C = 0$, in which case the big O will turn into a small o and we are done. It is sufficient to show $C = 0$ from one direction, say $\lambda \in \sigma_+$, which can be done using the scattering relations for θ_- as follows.

If $E \in \sigma_v$ this follows directly from

$$\theta_-(\lambda) = \frac{\overline{\phi_+} + R_+(\lambda)\phi_+(\lambda)}{T_+(\lambda)} = O\left(\frac{\hat{W}(\lambda)\hat{\phi}_+(\lambda)}{\hat{\delta}_-(\lambda)\sqrt{\lambda - E}}\right) = \begin{cases} o\left(\frac{1}{\lambda - E}\right), & E \in \hat{M}_-, \\ o\left(\frac{1}{\sqrt{\lambda - E}}\right), & E \notin \hat{M}_-. \end{cases}$$

Otherwise, if $E \notin \sigma_v$, then we have two representations

$$\theta_-(\lambda) = \frac{1}{T_+(\lambda)}\Big((\overline{\phi_+(\lambda)} - \phi_+(\lambda)) + \phi_+(\lambda)(R_+(\lambda) + 1)\Big), \quad \lambda \notin \hat{M}_+, \tag{5.47}$$

$$\theta_-(\lambda) = \frac{1}{T_+(\lambda)}\Big((\overline{\phi_+(\lambda)} + \phi_+(\lambda)) + \phi_+(\lambda)(R_+(\lambda) - 1)\Big), \quad \lambda \in \hat{M}_+. \tag{5.48}$$

For (5.47) we use $\overline{\phi_+(\lambda)} - \phi_+(\lambda) = o(1)$ by Lemma 3.1 (ii) and $R_+(\lambda) + 1 = o(1)$ by condition **III**. Now one checks

$$\left|\frac{1}{T_+(\lambda)}\right| + \left|\frac{\phi_+(z)}{T_+(z)}\right| = \begin{cases} O\left(\frac{1}{z-E}\right), & E \in \hat{M}_-, \\ O\left(\frac{1}{\sqrt{z-E}}\right), & E \notin \hat{M}_-, \end{cases} \quad \lambda \notin \hat{M}_+,$$

which shows $G(z,x) = o(\frac{1}{z-E})$ for $\lambda \notin \hat{M}_+$. For (5.48) we use $\overline{\phi_+(\lambda)} + \phi_+(\lambda) = O(1)$ and $R_+(\lambda) - 1 = o(1)$ as $\lambda \in \hat{M}_+$. Since in this case $\frac{1}{T_+(z)} = O\left(\frac{1}{\sqrt{z-E}}\right)$ and

$$\frac{\phi_+(z)}{T_+(z)} = \begin{cases} O\left(\frac{1}{z-E}\right), & E \in \hat{M}_-, \\ O\left(\frac{1}{\sqrt{z-E}}\right), & E \notin \hat{M}_-, \end{cases} \quad \lambda \notin \hat{M}_+,$$

this implies $G(z,x) = o(\frac{1}{z-E})$ for $\lambda \in \hat{M}_+$ as required. This finishes the case $E \in \Omega_3$.

• Ω_4: Finally we have to check, that the singularities at the points of the discrete spectrum are also removable. Since $\tilde{W}(z)$ has simple zeros at points λ_k, it suffices to check that

$$\tilde{\theta}_+(\lambda_k, x)\tilde{\theta}_-(\lambda_k, x) = \tilde{\phi}_-(\lambda_k, x)\tilde{\phi}_+(\lambda_k, x). \tag{5.49}$$

Lemma 5.4 shows, that $\tilde{\theta}_\mp := \delta_\mp\theta_\mp$, defined by (5.17), are continuous at the points $\breve{M}_\pm$. Since $(\delta_\mp T_\pm^{-1})(\lambda_k) = 0$ and $(\delta_\mp T_\pm^{-1}\breve{\psi}_\pm)(\lambda_k) = 0$, then the only the last summand in (5.17) is non-zero. Computing the limit of this summand at λ_k and using (3.10), (3.14) we obtain

$$\tilde{\theta}_\mp(\lambda_k) = \frac{d\tilde{W}(\lambda_k)}{d\lambda}(\gamma_\pm)^2\tilde{\phi}_\pm(\lambda_k), \tag{5.50}$$

which together with (3.16) implies (5.49). □

Lemma 5.7 implies, that $G(z,x)$ is an entire function for fixed x. Since in addition $G(z,x) \to 0$ as $z \to \infty$, Liouville's theorem implies $G(z,x) \equiv 0$ for every x. Therefore we have the equalities

$$\phi_+(z,x)\phi_-(z,x) = \theta_+(z,x)\theta_-(z,x) \tag{5.51}$$

and

$$\phi_\pm(\lambda,x)\overline{\theta_\pm(\lambda,x)} = \overline{\phi_\pm(\lambda,x)}\theta_\pm(\lambda,x), \quad \lambda \in \sigma^{(2)}. \tag{5.52}$$

It remains to show that $\theta_\pm(z,x) = \phi_\pm(z,x)$. This is equivalent to showing that

$$p(z,x) := \frac{\theta_+(z,x)}{\phi_+(z,x)} = \frac{\phi_-(z,x)}{\theta_-(z,x)}.$$

is equal to one. This function is well defined as long as $\hat\phi_+(z,x)\neq 0$. If $\hat\phi_+(z,x)=0$ this is still true as long as $\hat\theta_-(z,x)$ has no singularity (which is the case for $z\notin\partial\sigma_-$ by Lemma 5.4) and does not vanish. But for $z\notin\partial\sigma_-$ the case $\hat\phi_+(z,x)=\hat\theta_-(z,x)=0$ implies $\hat W(z)=0$, that is, $z\in\sigma_d\cup\partial\sigma_+$. Hence we will avoid such cases and suppose $x\notin X$, where

$$X:=\bigcup_{\lambda\in\sigma_d\cup\partial\sigma_-\cup\partial\sigma_+}\{x \mid \hat\phi_+(\lambda,x)=0\}.$$

Fix $x\in\mathbb{R}\setminus X$. Our aim is to show, that

$$p(z,x)\equiv 1,\quad z\in\mathbb{C},\ x\notin X. \tag{5.53}$$

By Lemma 3.1 and Corollary 5.6 the functions $\tilde\phi_\pm(z,x)$ and $\tilde\theta_\pm(z,x)$ are holomorphic in $\mathbb{C}\setminus\sigma_\pm$ and hence $p(z,x)$ is holomorphic on $\mathbb{C}\setminus\sigma^{(2)}$ with continuous limits up to the boundary away from $\partial\sigma^{(2)}$. By (5.52) the limits from different sides match up and so $p(z,x)$ is even holomorphic on $\mathbb{C}\setminus\partial\sigma^{(2)}$. Furthermore, arguing as before, one sees

$$\frac{\theta_+(z,x)}{\phi_+(z,x)}=\frac{\hat\theta_+(z,x)}{\hat\phi_+(z,x)}=O\Big(\frac{1}{\sqrt{z-E}}\Big),\quad E\in\partial\sigma_+,$$

that is, the singularities near $E\in\partial\sigma^{(2)}$ are removable and so $p(z,x)$ is entire with respect to z for all $x\notin X$. Finally, $p(z,x)\to 1$ as $z\to\infty$ by item (iii) of Lemma 2.1 respectively Lemma 5.4. In summary, (5.53) holds, that is,

$$\theta_\pm(z,x)\equiv\phi_\pm(z,x) \tag{5.54}$$

for all $x\notin X$. But since the set X is discrete, this even holds for all $x\in\mathbb{R}$ by continuity with respect to x.

Finally, (5.16) shows that $\tilde H_\pm\theta_\mp(z,x)=z\theta_\mp(z,x)$, that is, $\tilde q_+(x)\equiv q_-(x)$. Moreover, (5.16) and (5.54) imply, that

$$T_\mp(\lambda)\phi_\pm(\lambda,x)=\overline{\phi_\mp(\lambda,x)}+R_\mp(\lambda)\phi_\mp(\lambda,x),$$

and from (5.50), (5.54), (3.16), and (3.21) we conclude that (3.11) is valid. Therefore the data $\mathcal{S}$ are the scattering data for the Schrödinger operator with the potential $q(x)=\tilde q_-(x)=\tilde q_+(x)$. This finishes the proof of Theorem 5.3. □

Finally, observe that the second moment in condition **IV** from Lemma 4.2 can be replaced by any higher moment. In fact, introduce

IV*. *The function $F_\pm(x,y)$ is continuously differentiable with respect to both variables and there exists real-valued function $q_\pm(x)$, $x\in\mathbb{R}$, with*

$$\pm\int_a^{\pm\infty}(1+|x|^n)|q_\pm(x)|dx<\infty,\quad \text{for all } a\in\mathbb{R},$$

such that

$$\begin{aligned}&|F_\pm(x,y)|\le C_\pm(x)Q_\pm(x+y),\\ &\left|\frac{\partial}{\partial x}F_\pm(x,y)\right|\le C_\pm(x)\left(\left|q_\pm\left(\frac{x+y}{2}\right)\right|+Q_\pm(x+y)\right),\\ &\pm\int_a^{\pm\infty}\left|\frac{d}{dx}F_\pm(x,x)\right|(1+|x|^n)\,dx<\infty,\end{aligned}$$

where $n = 2, 3, 4, \dots,$

$$Q_\pm(x) := \pm \int_{\frac{x}{2}}^{\pm\infty} |q_\pm(t)|\, dt,$$

and $C_\pm(x) > 0$ *is a continuous function, which decreases monotonically as* $x \to \pm\infty$.

Then, proceeding literally as in Theorem 5.3, we obtain

Theorem 5.8. *Let the scattering data* $\mathcal{S}$*, defined as in* (4.22)*, satisfy conditions* **I–III***, and* **IV****. Then each of the GLM equations* (4.1) *has a unique solution* $K_\pm(x, y)$*. The functions* $\tilde{q}_\pm(x)$ *defined by* (5.7)*, coincide:*

$$\tilde{q}_-(x) = \tilde{q}_+(x) = q(x)$$

and satisfy

$$\pm \int_a^{\pm\infty} |q(x) - p_\pm(x)|(1 + |x|^n) dx < \infty, \quad a \in \mathbb{R}. \tag{5.55}$$

6. The Korteweg–de Vries equation with steplike finite-gap initial data

In this final section we will outline in what way the inverse scattering transform method can be used to study the initial-value problem for the Korteweg–de Vries equation with steplike finite-gap initial data. Note that the Cauchy problem for steplike constant initial data is studied in [5, 20]. In the case of a constant background on the right half-axis and periodic finite-gap background on the left one, this problem is partly considered in [12]. The existence of the solution of the KdV equation with general finite-gap steplike potential as initial data seems to be an open problem and is not a subject of the present paper. Here we only discuss a possible approach.

Let $q(x)$ be a smooth function, satisfying condition (5.55) for $n = 5$ together with its derivatives:

$$\pm \int_a^{\pm\infty} |q^{(k)}(x) - p_\pm^{(k)}(x)|(1 + |x|^5) dx < \infty, \quad a \in \mathbb{R}, \quad k = 0, 1, \dots, 7. \tag{6.1}$$

Here $p_\pm(x)$ are some finite-gap potentials.

Consider the initial value problem

$$\frac{\partial u}{\partial t} - 6u\, \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{6.2}$$

$$u(x, 0) = q(x), \quad x \in \mathbb{R}. \tag{6.3}$$

One can ask to find a unique smooth solution of this problem in the domain $|t| < T$, satisfying conditions

$$\sup_{|t|<T} \pm \int_0^{\pm\infty} (1 + |x|^2)|u(x, t) - u_\pm(x, t)| dx < \infty, \tag{6.4}$$

$$\sup_{|t|<T} \pm \int_0^{\pm\infty} (1 + |x|) \left| \frac{\partial^k u(x, t)}{\partial x^k} - \frac{\partial^k u_\pm(x, t)}{\partial x^k} \right| dx < \infty, \quad k = 1, 2, 3, \tag{6.5}$$

where the functions $u_\pm(x, t)$ are the finite-gap solutions of equation (6.2) with initial data $u_\pm(x, 0) = p_\pm(x)$. We will proceed by a standard scheme

$$u(x, 0) \rightsquigarrow \mathcal{S}(0) \rightsquigarrow \mathcal{S}(t) \rightsquigarrow u(x, t).$$

The Lax pair, associated with the KdV equation has the form

$$P(t) = -4\frac{\partial^3}{\partial x^3} + 6u(x,t)\frac{\partial}{\partial x} + 3u_x(x,t), \tag{6.6}$$

$$H(t) = -\frac{\partial^2}{\partial x^2} + u(x,t). \tag{6.7}$$

Equation (6.2) is then equivalent to equation $\partial_t H = [H, P]$. Let $H_\pm(t)$, $P_\pm(t)$ be Lax pairs, corresponding to our backgrounds. Following the scheme proposed in [9], we will obtain the time-dependent GLM equation:

Lemma 6.1. *Let $\psi_\pm(z,x,t)$ be the Weyl solutions satisfying (see, e.g., [17])*

$$H_\pm(t)\psi = z\psi, \qquad P_\pm(t)\psi = \frac{\partial}{\partial t}\psi, \qquad \psi_\pm(z,0,0) = 1. \tag{6.8}$$

Then the inverse scattering problem (the GLM) equation has the form

$$K_\pm(x,y,t) + F_\pm(x,y,t) \pm \int_x^{\pm\infty} K_\pm(x,s,t)F_\pm(s,y,t)ds = 0, \quad \pm y > \pm x, \tag{6.9}$$

where

$$\begin{aligned} F_\pm(x,y,t) = &\oint_{\sigma_\pm} R_\pm(\lambda,0)\,\psi_\pm(\lambda,x,t)\psi_\pm(\lambda,y,t)d\rho_\pm(\lambda,0) \\ &+ \int_{\sigma_\mp^{(1),\mathrm{u}}} |T_\mp(\lambda,0)|^2\psi_\pm(\lambda,x,t)\psi_\pm(\lambda,y,t)d\rho_\mp(\lambda,0) \\ &+ \sum_{k=1}^{p} \gamma_k^\pm(0)^2\tilde\psi_\pm(\lambda_k,x,t)\tilde\psi_\pm(\lambda_k,y,t). \end{aligned} \tag{6.10}$$

To prove that the problem (6.2)–(6.3) has a solution in the class (6.1)–(6.5) one has to check, that time-dependent scattering data satisfy conditions of Theorem 5.3. Conditions **I**, **II**, **III (a)** can be verified directly from the time evolution for scattering data

$$\begin{aligned} R_\pm(\lambda,t) &= R_\pm(\lambda,0)\exp\big(\alpha_\pm(\lambda,t) - \overline{\alpha_\pm(\lambda,t)}\big), \\ T_\pm(\lambda,t) &= T_\pm(\lambda,0)\exp\big(\overline{\alpha_\pm(\lambda,t)} - \alpha_\mp(\lambda,t)\big), \\ \gamma_k^\pm(t)^2 &= \gamma_k^\pm(0)^2\exp(2\alpha_\pm(\lambda_k,t)), \end{aligned}$$

where (cf. [12, 16])

$$\alpha_\pm(\lambda,t) = \int_0^t \left(2(u_\pm(0,s)+2\lambda)m_\pm(\lambda,s) - \frac{\partial u_\pm(0,s)}{\partial x}\right)ds$$

and $m_\pm(\lambda,t)$ are time-dependent Weyl functions, corresponding to background operators $H_\pm(t)$. Note that condition **III, (b)** means, that when $E \notin \sigma_v$ and $\mu_j^\pm(t)$ gets close to E, then $R_\pm(E,t)$ changes its sign. This effect was explained in [9].

To check condition **IV** one has to take into account the structure of the Weyl solutions $\psi_\pm(\lambda,x,t)$. As is known (see e.g., [1]), they admit a representation $\psi_\pm(\lambda,x,t) = \exp(\pm\mathrm{i}k_\pm(\lambda)x)f_\pm(\lambda,x,t)$, where $k_\pm(\lambda)$ are the quasi-momentum maps. To estimate the parts $F_{d,\pm}(x,y,t)$ and $F_{h,\pm}(x,y,t)$ of the kernel $F_\pm$ (cf. (4.9), (4.15), (4.17) it is sufficient to use the Herglotz property of the quasi-momentum ($\mathrm{Im}(k_\pm(\lambda)) > 0$ as $\lambda \in \mathbb{R}\setminus\sigma_\pm$) and the fact, that the functions $f_\pm(\lambda,x,t)$ are bounded as $x\in\mathbb{R}$, $|t|<T$ and $\lambda \notin M_\pm(0)\cup\hat M_\pm(0)$, $\lambda > \inf\{E_0^+, E_0^-, \lambda_1\} - 1$. To estimate $F_{r,\pm}(x,y,t)$ we integrate the first summand in (6.10) twice by parts with respect to the quasi-momentum variable $k_\pm$ and prove, that the boundary terms vanish. This approach fails only in points of the set $(\partial\sigma_-^{(1)}\cup\partial\sigma_+^{(1)})\cap\partial\sigma^{(2)}$ (the points of type E_5 in our example). Here one has to use the approach, developed in [20, Proposition 2.7]. This way one arrives at the estimates

$$|F_\pm(x,y,t)|\leq\frac{C(t)}{|x+y|^3},\quad\left|\frac{\partial F_\pm(x,y,t)}{\partial x}\right|\leq\frac{C(t)}{|x+y|^4},\quad x,y\to\pm\infty,$$

that justify condition (6.4).

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Appendix A. Properties of the transformation operators and estimates for the GLM kernel

In this appendix we derive and thoroughly investigate the integral equations for the kernels $K_\pm(x,y)$ of the transformation operators. In addition, we will obtain the necessary estimates for them and their derivatives. This will allow us to simplify the necessary and sufficient conditions on the functions $F_\pm(x,y)$ (in comparison with [14]) and to solve the scattering problem in the prescribed class of perturbations (1.3).

Most of the results from this section are in essence known or follow as in the case of a constant background (see, e.g., [13, 14], and in the discrete case [4, 8, 21]). We included them here (with proofs) to make our presentation self-contained.

Let $\psi_\pm(z,x)$ be the background Weyl solution (2.9). Set

$$J_\pm(z,x,y)=\frac{\psi_\pm(z,y)\breve{\psi}_\pm(z,x)-\psi_\pm(z,x)\breve{\psi}_\pm(z,y)}{W(\psi_\pm(z),\breve{\psi}_\pm(z))}\tag{A.1}$$

and

$$q_\pm(x)=q(x)-p_\pm(x).\tag{A.2}$$

Then the Jost solutions (3.2) satisfy the integral equation

$$\phi_\pm(z,x)=\psi_\pm(z,x)-\int_x^{\pm\infty}J_\pm(z,x,y)q_\pm(y)\phi_\pm(z,y)dy.\tag{A.3}$$

If we substitute formula (3.2) into this equation, multiply with $\breve{\psi}_\pm(z,s)g_\pm(z)$, and integrate over the set $\sigma_\pm^{\mathrm{u,l}}$, using the inverse Fourier transform (5.11), and taking into account that $K_\pm(x,y)=0$, $\pm x>\pm y$, we obtain

$$\begin{aligned}K_\pm(x,s)&+\int_x^{\pm\infty}dy\,q_\pm(y)\oint_{\sigma_\pm}J_\pm(\lambda,x,y)\psi_\pm(\lambda,y)\breve{\psi}_\pm(\lambda,s)d\rho_\pm(\lambda)\\ &\pm\int_x^{\pm\infty}dy\,q_\pm(y)\int_y^{\pm\infty}dt\,K_\pm(y,t)\oint_{\sigma_\pm}J_\pm(\lambda,x,y)\psi_\pm(\lambda,t)\breve{\psi}_\pm(\lambda,s)d\rho_\pm(\lambda)=0.\end{aligned}\tag{A.4}$$

Set

$$\Gamma_\pm(x,y,t,s)=\mp\oint_{\sigma_\pm}\psi_\pm(\lambda,x)\breve{\psi}_\pm(\lambda,y)\psi_\pm(\lambda,t)\breve{\psi}_\pm(\lambda,s)g_\pm(\lambda)d\rho_\pm(\lambda),\tag{A.5}$$

where the integral has to be understood as a principal value.

Then substituting (2.14), (2.11), (A.1), and (A.5) into (A.4) we obtain

$$K_\pm(x,s) + \int_x^{\pm\infty} \left(\Gamma_\pm(x,y,y,s) - \Gamma_\pm(y,x,y,s)\right) q_\pm(y)\, dy \\ \pm \int_x^{\pm\infty} dy\, q_\pm(y) \int_y^{\pm\infty} K_\pm(y,t) \left(\Gamma_\pm(x,y,t,s) - \Gamma_\pm(y,x,t,s)\right) dt = 0. \tag{A.6}$$

Consider now the function in (A.5). From (2.13) it follows that

$$\overline{\Gamma_\pm(x,y,t,s)} = -\Gamma_\pm(y,x,s,t). \tag{A.7}$$

Our plan is to evaluate the integral in (A.5) using the Jordan lemma. The only poles of the integrand in (A.5) are at the band edges and hence we introduce

$$f_\pm(E,x,y) = \lim_{z\to E} \Big(\prod_{j=1}^{r_\pm} (z - \mu_j^\pm) \Big) \psi_\pm(z,x) \breve{\psi}_\pm(z,y) \tag{A.8}$$

and

$$D_\pm(x,y,t,s) = \pm\frac{1}{4} \sum_{E\in\partial\sigma_\pm} \frac{f_\pm(E,x,y) f_\pm(E,t,s)}{\frac{d}{dz} P_\pm(E)}, \quad P_\pm(z) = \prod_{j=0}^{2r_\pm} (z - E_j^\pm). \tag{A.9}$$

Note that $D_\pm(x,y,t,s)$ is a continuous and bounded function with respect to all variables.

Now suppose $\pm(x-y+t-s) > 0$ and take a closed contour consisting of a large circular arc together with some parts wrapping around the spectrum $\sigma_\pm$ inside this arc at a small distance from the spectrum. Due to (2.11), (2.7), and (iii) of Lemma 2.1, it follows that

$$g_\pm(z)^2 \psi_\pm(z,x) \breve{\psi}_\pm(z,y) \psi_\pm(z,t) \breve{\psi}_\pm(z,s) = O\left(\frac{1}{z}\right) \mathrm{e}^{\pm \mathrm{i}\sqrt{z}(x-y+t-s)}$$

as $z \to \infty$. In fact this holds on the entire circle since the neighborhood of the positive real axis can be handled as above. Hence one can apply Jordan's lemma to conclude that the contribution of the circle vanishes as its radius tends to infinity. Shrinking the loops the integral converges to

$$\Gamma_\pm(x,y,t,s) = D_\pm(x,y,t,s), \quad \text{for} \quad \pm(x-y+t-s) > 0. \tag{A.10}$$

Note that $f_\pm(E,x,y)$ are real, and $f_\pm(E,x,y) = f_\pm(E,y,x)$. Thus, the $D_\pm(x,y,t,s)$ is also real,

$$D_\pm(x,y,t,s) = D_\pm(y,x,t,s). \tag{A.11}$$

Now let $\pm(x-y+t-s) < 0$, that is, $\pm(y-x+s-t) > 0$. Then (A.7), (A.10), and (A.11) imply

$$\Gamma_\pm(x,y,t,s) = -\overline{D_\pm(x,y,t,s)} = -D_\pm(x,y,t,s), \quad \pm(x-y+t-s) < 0.$$

Therefore,

$$\Gamma_\pm(x,y,t,s) = D_\pm(x,y,t,s)\, \mathrm{sign}(\pm(x-y+t-s)). \tag{A.12}$$

Property (A.11) implies that the domain, where in the first integrand in (A.6) does not vanish, is

$$\mathrm{sign}(\pm(x-s)) = -\,\mathrm{sign}(\pm(2y-x-s)), \quad \pm s > \pm x. \tag{A.13}$$

In the second integral the domain of integration is

$$\begin{aligned} &\operatorname{sign}(\pm(x-y+t-s)) = -\operatorname{sign}(\pm(y-x+t-s)),\\ &\text{with} \quad \pm s > \pm x, \quad \pm t > \pm y > \pm x. \end{aligned} \tag{A.14}$$

Solving (A.13) and (A.14) we arrive at the following result.

Lemma A.1. *The kernels* $K_\pm(x,s)$ *of the transformation operators satisfy the integral equation*

$$\begin{aligned} K_\pm(x,s) &= -2\int_{\frac{x+s}{2}}^{\pm\infty} q_\pm(y) D_\pm(x,y,y,s) dy \\ &\mp 2\int_x^{\pm\infty} dy \int_{s\pm x\mp y}^{s\pm y\mp x} D_\pm(x,y,t,s) K_\pm(y,t) q_\pm(y)\, dt, \quad \pm s > \pm x, \end{aligned} \tag{A.15}$$

where $D_\pm$ *are defined by* (A.9).

Set $s=x$ in (A.15). Then the second summand vanishes, because we have our integration inside the domain $\pm t < \pm y$, where $K_\pm(y,t)=0$. Thus

$$K_\pm(x,x) = -2\int_x^{\pm\infty} q_\pm(y) D_\pm(x,y,y,x)\, dy. \tag{A.16}$$

But, as is well-known (see, e.g, [17, Eq. (1.84)] or [25, Chapter 8])

$$\psi_\pm(z,y)\breve{\psi}_\pm(z,y)\prod_{j=1}^{r_\pm}(z-\mu_j^\pm) = \prod_{j=1}^{r_\pm}(z-\mu_j^\pm(y)), \tag{A.17}$$

where $\mu_j^\pm(y)$ are the Dirichlet eigenvalues corresponding to the base point $x=y$ (rather than $x=0$). Combining (A.5) and (A.10) we obtain

$$\begin{aligned} D_\pm(x,y,y,x) &= D_\pm(x,x,y,y) \\ &= \pm\frac{1}{4}\sum_{E\in\partial\sigma_\pm} \operatorname{Res}_E \frac{\prod_{j=1}^{r_\pm}\big(z-\mu_j^\pm(x)\big)\big(z-\mu_j^\pm(y)\big)}{(z-E_0^\pm)\prod_{j=1}^{r_\pm}\big((z-E_{2j-1}^\pm)(z-E_{2j}^\pm)\big)}. \end{aligned} \tag{A.18}$$

The integrand in (A.18) is meromorphic in $\mathbb{C}$, thus, by the Cauchy theorem, we can compute the residue at infinity and obtain

$$D_\pm(x,y,y,x) = -\lim_{z\to\infty} \pm\frac{1}{4}\frac{z}{z-E_0^\pm}\prod_{j=1}^{r_\pm}\frac{\big(z-\mu_j^\pm(y)\big)\big(z-\mu_j^\pm(x)\big)}{(z-E_{2j-1}^\pm)(z-E_{2j}^\pm)} = \mp\frac{1}{4}.$$

From (A.16) we conclude that

$$K_\pm(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty}(q(t)-p_\pm(t))dt. \tag{A.19}$$

This formula justifies formula (5.7) under the condition, that the transformation operators kernels are differentiable.

Lemma A.2. *Let*

$$Q_\pm(x) := \pm\int_{\frac{x}{2}}^{\pm\infty} |q_\pm(t)|\, dt, \quad q_\pm(x) = q(x) - p_\pm(x). \tag{A.20}$$

Then $K_\pm(x,y)$ has first order partial derivatives with respect to both variables. Moreover, for $\pm y \ge \pm x$ the following estimates are valid

$$|K_\pm(x,y)| \le C_\pm(x) Q_\pm(x+y), \tag{A.21}$$

$$\left|\frac{\partial K_\pm(x,y)}{\partial x}\right| + \left|\frac{\partial K_\pm(x,y)}{\partial y}\right| \le C_\pm(x)\left(\left|q_\pm\left(\frac{x+y}{2}\right)\right| + Q_\pm(x+y)\right), \tag{A.22}$$

where $C_\pm(x)$ are positive continuous functions for $x\in\mathbb{R}$ which decrease as $x\to\pm\infty$ and depend on the corresponding background data and on the first moment of the perturbation.

Proof. We restrict our considerations to the "+" case only and omit "+" in what follows. We will follow the scheme of the proof of [26, Lemmas 3.1.1, 3.1.2]. Introduce the following change of variables in (A.15):

$$y+t=:2\alpha,\quad t-y=:2\beta,\quad x+s=:2u,\quad s-x=:2v, \tag{A.23}$$

then from (A.15) we obtain (see [26, Lemma 3.1.1])

$$\begin{aligned} H(u,v) = &-2\int_u^\infty q(s)D_1(u,v,s)\,ds \\ &- 4\int_u^\infty d\alpha \int_0^v q(\alpha-\beta)D_2(u,v,\alpha,\beta)H(\alpha,\beta)\,d\beta, \end{aligned} \tag{A.24}$$

where we put

$$\begin{aligned} &H(u,v)=K_+(u-v,u+v),\quad D_1(u,v,s)=D_+(u-v,s,s,u+v),\\ &D_2(u,v,\alpha,\beta)=D_+(u-v,\alpha-\beta,\alpha+\beta,u+v). \end{aligned} \tag{A.25}$$

Functions D_1 and D_2 are bounded uniformly with respect to all their variables. Put $C=2\max\{\max_{u,v,s}|D_1|,\max_{u,v,\alpha,\beta}|D_2|\}$ and apply the method of successive approximations (see [26, Lemma 3.1.1]). We arrive at the estimate

$$|H(u,v)| \le \tilde C(u-v)\int_u^\infty |\tilde q(x)|\,dx, \tag{A.26}$$

with

$$\tilde C(u) = C\exp\left(C\int_{2u}^\infty Q(2t)dt\right),\quad C>0, \tag{A.27}$$

from which (A.21) follows. To obtain (A.22), observe that the first partial derivatives of D_1 and D_2 exist (see (A.8), (A.9) and (A.25)) and are bounded with respect to all variables. Thus,

$$\begin{aligned} &\frac{\partial H(u,v)}{\partial u} - 2q(u)D_1(u,v,u) = \\ &= -4\int_0^v q(u-\beta)D_2(u,v,u,\beta)H(u,\beta)\,d\beta - 2\int_u^\infty q(s)\frac{\partial D_1(u,v,s)}{\partial u}\,ds \\ &\quad - 4\int_u^\infty d\alpha\int_0^v q(\alpha-\beta)\frac{\partial D_2(u,v,\alpha,\beta)}{\partial u}H(\alpha,\beta)\,d\beta, \end{aligned} \tag{A.28}$$

$$\frac{\partial H(u,v)}{\partial v} = \\ = -2\left(\int_u^\infty q(s)\frac{\partial D_1(u,v,s)}{\partial v}\,ds - 2\int_u^\infty q(\alpha-v)D_2(u,v,\alpha,v)H(\alpha,v)\,d\alpha \right. \\ \left. -2\int_u^\infty d\alpha\int_0^v q(\alpha-\beta)\frac{\partial D_2(u,v,\alpha,\beta)}{\partial v}H(\alpha,\beta)\,d\beta\right). \tag{A.29}$$

The function $Q(u)=\int_u^\infty |q(x)|\,dx$ is positive, monotonically decreasing, and satisfies $Q(\,\cdot\,)\in L^1(a,\infty)$, $a\in\mathbb{R}$. Since $|D_i|$, $|\frac{\partial D_i}{\partial u}|$, $|\frac{\partial D_i}{\partial v}|\le C_1$, $i=1,2$, (A.26) applied to (A.28) and (A.29) implies

$$\left|\frac{\partial H(u,v)}{\partial u} - 2q(u)D_1(u,v,u)\right| + \left|\frac{\partial H(u,v)}{\partial v}\right| \le \tilde C(u-v)Q(2u),$$

where the function $\tilde C(u)$ is of the same type as (A.27), with a different positive constant C_2 depending on the background data. From this, (A.23), and (A.25), the estimate (A.22) follows. □

With the help of this lemma we can now derive several estimates for the GLM equation.

Lemma A.3. *The kernel $F_\pm(x,y)$ of the GLM equation* (4.1) *has first order derivatives with respect to each variable. Furthermore, for $\pm y>\pm x$ it satisfies*

$$|F_\pm(x,y)|\le \hat C_\pm(x)\,Q_\pm(x+y), \tag{A.30}$$

$$\left|\frac{\partial F_\pm(x,y)}{\partial x}\right| + \left|\frac{\partial F_\pm(x,y)}{\partial y}\right| \le \hat C_\pm(x)\,\left(\left|q_\pm\left(\frac{x+y}{2}\right)\right| + Q_\pm\,(x+y)\right), \tag{A.31}$$

where the functions $q_\pm(x)$ and $Q_\pm(x)$ are defined in (A.20). *Here $\hat C_\pm(x)$ are positive continuous functions which decrease as $x\to\pm\infty$. Moreover,*

$$\pm\int_a^{\pm\infty}(1+x^2)\left|\frac{dF_\pm(x,x)}{dx}\right| < \infty, \quad \forall a\in\mathbb{R}. \tag{A.32}$$

Proof. Again we restrict our considerations to the "+" case only and omit "+" in what follows. Set $Q_1(u)=\int_u^\infty Q(t)dt$. Due to condition (1.3), the functions $Q(x)$ and $Q_1(x)$ satisfy

$$\int_a^\infty Q_1(t)dt<\infty, \quad \int_a^\infty Q(t)(1+|t|)dt<\infty. \tag{A.33}$$

Observe also, that the kernel $F(x,y)$ of the GLM equation (4.1) is symmetric: $F(x,y)=F(y,x)$. From (4.1) and (A.21) we see, that

$$|F(x,y)|\le \tilde C(x)\left(Q(x+y)+\int_x^\infty Q(x+t)|F(t,y)|dt\right).$$

Since $Q_1(x+t)>Q_1(2x)$, and $\tilde C(t)<\tilde C(x)$ as $x<t$, then Gronwall's inequality implies (A.30) with

$$\hat C(x)=C_1\tilde C(x)\exp\bigl(C_1\tilde C(x)Q_1(2x)\bigr), \quad C_1>0.$$

Differentiating (4.1) with respect to x and y implies

$$|F_x(x,y)| \le |K_x(x,y)| + |K(x,x)F(x,y)| + \int_x^\infty |K_x(x,t)F(t,y)|\,dt, \tag{A.34}$$

$$F_y(x,y) + K_y(x,y) + \int_x^\infty K(x,t)F_y(t,y)dt = 0. \tag{A.35}$$

The functions $Q(x)$, $Q_1(x)$, $\hat{C}(x)$, $\tilde{C}(x)$ are monotonously decreasing and positive. Furthermore,

$$\int_x^\infty \left(\left|q_\pm\left(\frac{x+t}{2}\right)\right| + Q(x+t)\right) Q(t+y)dt \le (Q(2x)+Q_1(2x))Q(x+y),$$

and hence the estimate (A.31) for F_x follows (with some other positive continuous decreasing function $\hat{C}(x)$) from (A.30), (A.21), (A.22) and (A.34). The same estimate for F_y can be obtained from (A.35) and Lemma A.2 by using the method of successive approximations.

It remains to prove (A.32). To this end consider (4.1) for $y=x$ and differentiate it with respect to x:

$$\frac{dF(x,x)}{dx} + \frac{dK(x,x)}{dx} - K(x,x)F(x,x) + \int_x^\infty \left(K_x(x,t)F(t,x) + K(x,t)F_y(t,x)\right)dt = 0.$$

Formula (A.19) implies (5.2). Next, by (A.30) and (A.21), we have

$$|K(x,x)F(x,x)| \le \tilde{C}(a)\hat{C}(a)Q^2(2x) \text{ for } x > a,$$

where $\int_a^\infty (1+x^2)Q^2(2x)dx < \infty$. Moreover, by (A.31) and (A.22),

$$|K_x'(x,t)F(t,x)| + \left|K(x,t)F_y'(t,x)\right| \le 4\tilde{C}(a)\hat{C}(a)\Big\{\Big|q\Big(\frac{x+t}{2}\Big)\Big|Q(x+t) + Q^2(x+t)\Big\},$$

and together with the estimates

$$\int_a^\infty dx\, x^2 \int_x^\infty Q^2(x+t)dt \le \int_a^\infty |x|Q(2x)dx \sup_{x\ge a}\int_x^\infty |x+t|Q(x+t)dt < \infty,$$

$$\int_a^\infty x^2 \int_x^\infty \Big|q\Big(\frac{x+t}{2}\Big)\Big|Q(x+t)dt \le$$
$$\le \int_a^\infty Q(2x)dx \sup_{x\ge a}\int_x^\infty \Big|q\Big(\frac{x+t}{2}\Big)\Big|(1+(x+t)^2)dt < \infty$$

we arrive at (A.32). □

Remark A.4. Note that the results of this lemma are in some sense invertible. Namely, if we start with properties (4.18)–(4.21) of $F_\pm(x,y)$, then, using (4.1) and the same considerations as in Lemma A.3, we obtain (5.1), (A.22), and (5.2).

Appendix B. Proof of Lemma B.4

We introduce the local parameter $\tau = \sqrt{z-E}$ in a small vicinity of each point $E \in \partial\sigma_\pm$ and set $\dot{y}(z,x) = \frac{\partial}{\partial\tau}y(z,x)$. Since $\frac{dz}{d\tau}(E) = 0$, for every solution $y(z,x)$ of the equation (2.1), its derivative $\dot{y}(E,x)$ is also a solution of (2.1). In particular, the Wronskian $W(y(E),\dot{y}(E))$ is independent of x.

For each $x \in \mathbb{R}$ in a small neighborhood of a point $E \in \partial\sigma_\pm$ introduce the function

$$\hat{\psi}_{\pm,E}(z,x) = \begin{cases} \psi_\pm(z,x), & E \in \partial\sigma_\pm \setminus \hat{M}_\pm, \\ \tau\, \psi_\pm(z,x), & E \in \hat{M}_\pm. \end{cases} \tag{B.1}$$

Lemma B.1. *Let the function* $\hat{\psi}_{\pm,E}(z,x)$ *be defined by formula* (B.1). *Then*

$$W\Big(\hat{\psi}_{\pm,E}(E), \frac{\partial}{\partial\tau}\hat{\psi}_{\pm,E}(E)\Big) = \pm \lim_{z\to E} \frac{\alpha\,\tau^\alpha}{2g_\pm(z)}, \tag{B.2}$$

where $\alpha = -1$ *if* $E \in \partial\sigma_\pm \setminus \hat{M}_\pm$ *and* $\alpha = 1$ *if* $E \in \hat{M}_\pm$.

Proof. We begin by recalling (see, e.g., [17, Eq. (1.73)]) that the Weyl m-functions can be written as

$$m_\pm(z) = \frac{G_\pm(z) \pm \sqrt{P_\pm(z)}}{F_\pm(z)}, \tag{B.3}$$

where $G_\pm(z) = \frac{1}{2}F'_\pm(z,0)$ and $F_\pm(z) = F_\pm(z,0)$ with

$$P_\pm(z) = \prod_{j=0}^{2r_\pm}(z - E_j^\pm), \qquad F_\pm(z,x) = \prod_{j=1}^{r_\pm}(z - \mu_j^\pm(x)). \tag{B.4}$$

Furthermore, observe that $\mp\sqrt{P_\pm(z)} = \tau f_\pm(z)$, where $f_\pm(z)$ is holomorphic near $z = E$.

Then in the case $E \in \partial\sigma_\pm \setminus \hat{M}_\pm$ (where $\hat{\psi}_{\pm,E}(\lambda,x) = \psi_\pm(\lambda,x)$) we have $F_\pm(E) \neq 0$. Then, from (2.9), we have $\psi_\pm(E,0) = 1$, $\psi'_\pm(E,0) = m_\pm(E)$, $\dot{\psi}_\pm(E,0) = 0$,

$$\dot{\psi}'_\pm(E,0) = \dot{m}_\pm(E) = \frac{f_\pm(E)}{F_\pm(E)} = \mp \lim_{z\to E} \frac{\sqrt{P_\pm(z)}}{\tau F_\pm(z)},$$

and the first claim follows.

In the second case we have $E \in \hat{M}_\pm$ (where $\hat{\psi}_{\pm,E}(\lambda,x) = \tau\psi_\pm(\lambda,x)$) we have $F_\pm(E) = 0$ and hence $F_\pm(z) = \tau^2\tilde{F}_\pm(z)$, $G_\pm(z) = \tau^2\tilde{G}_\pm(z)$, where $\tilde{F}_\pm(z)$ and $\tilde{G}_\pm(z)$ are holomorphic near $z = E$ with $\tilde{F}_\pm(E) \neq 0$ and $\tilde{G}_\pm(E) \neq 0$. Hence we have

$$\hat{\psi}_{\pm,E}(E,0) = 0, \qquad \hat{\psi}'_{\pm,E}(E,0) = \frac{f_\pm(E)}{\tilde{F}_\pm(E)},$$

$$\frac{\partial}{\partial\tau}\hat{\psi}_{\pm,E}(E,0) = 1, \qquad \frac{\partial}{\partial\tau}\hat{\psi}'_{\pm,E}(E,0) = \frac{\tilde{G}_\pm(E)}{\tilde{F}_\pm(E)}$$

and the second claim follows. □

From (2.5), (2.10) and (2.9) we observe the following property of the Floquet–Weyl solutions

Remark B.2. Let $E \in \hat{M}_\pm$ and $x \in \mathbb{R}$. Then $\overline{\hat{\psi}_{\pm,E}(E,x)} = -\hat{\psi}_{\pm,E}(E,x)$ if E is a left band edge from $\sigma_\pm$ and $\overline{\hat{\psi}_{\pm,E}(E,x)} = \hat{\psi}_{\pm,E}(E,x)$ if E is a right band edge.

It is straightforward to check that this property is also inherited by the Jost solutions. Again we abbreviate

$$\hat{\phi}_{\pm,E}(\lambda,x) = \begin{cases} \phi_\pm(\lambda,x), & E \in \partial\sigma_\pm \setminus \hat{M}_\pm, \\ \tau\, \phi_\pm(\lambda,x), & E \in \hat{M}_\pm. \end{cases} \tag{B.5}$$

Lemma B.3. *We have* $\overline{\phi_\pm(E,x)} = \phi_\pm(E,x)$ *for* $E \in \partial\sigma_\pm \setminus \hat M_\pm$. *If* $E \in \hat M_\pm$, *then by* (B.5) $\overline{\hat\phi_{\pm,E}(E,x)} = -\hat\phi_{\pm,E}(E,x)$ *when* E *is a left edge of a band from* $\sigma_\pm$ *and* $\overline{\hat\phi_{\pm,E}(E,x)} = \hat\phi_{\pm,E}(E,x)$ *when* E *is a right edge of a band.*

Lemma B.4. *The function* $\hat W(z)$ *is continuous on the set* $\mathbb{C}\setminus\sigma$ *up to the boundary* $\sigma^{\mathrm{u}} \cup \sigma^{\mathrm{l}}$. *It can have zeros on the set* $\partial\sigma \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)})$ *and does not vanish at the other points of the set* σ. *If* $\hat W(E) = 0$ *as* $E \in \partial\sigma \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)})$, *then* $\hat W(z) = \sqrt{z-E}(C(E) + o(1))$, $C(E) \neq 0$.

Proof. Continuity of $\hat W$ up to the boundary follows from the corresponding property of $\hat\phi_\pm(z,x)$. We begin with the investigation of the possible zeros.

Let $\lambda_0 \in \mathrm{int}(\sigma^{(2)}) := \sigma^{(2)} \setminus \partial\sigma^{(2)}$ and suppose $W(\lambda_0) = 0$. Then $\phi_+(\lambda_0,x) = c\,\phi_-(\lambda_0,x)$, $\overline{\phi_+(\lambda_0,x)} = \bar c\,\overline{\phi_-(\lambda_0,x)}$, i.e. $W(\phi_+,\overline{\phi_+}) = |c|^2 W(\phi_-,\overline{\phi_-})$. But by (2.11) and (2.14) we have $\operatorname{sign} g_+(\lambda_0) = -\operatorname{sign} g_-(\lambda_0)$, contradicting (2.6).

Let $\lambda_0 \in \mathrm{int}(\sigma_\pm^{(1)})$ and $\tilde W(\lambda_0) = 0$. The point λ_0 can coincide with a pole $\mu \in M_\mp$. But $\phi_\pm(\lambda_0,x)$ and $\overline{\phi_\pm(\lambda_0,x)}$ are linearly independent and bounded, and $\tilde\phi_\mp(x,\lambda_0) \in \mathbb{R}$ (see (3.7)). If $W(\lambda_0) = 0$, then $\tilde\phi_\mp = c_1^\pm\,\phi_\pm = c_2^\pm\,\overline{\phi_\pm}$, that implies $W(\phi_\pm,\overline{\phi_\pm})(\lambda_0) = 0$, which is impossible.

In the general mutual location of the background spectra one can meet the case when $\lambda_0 = E \in \partial\sigma^{(2)} \cap \mathrm{int}(\sigma_\pm) \subset \mathrm{int}(\sigma)$ (that is a point like E_5 in our example). If $\hat W(E) = 0$, then $W(\phi_\pm, \hat\phi_{\mp,E})(E) = 0$, where $\hat\phi_{\mp,E}$ is defined by (B.5). But according to Lemma B.3, the values of $\hat\phi_{\mp,E}(E,\cdot)$ are either pure real or pure imaginary, therefore $W(\overline{\phi_\pm}, \hat\phi_{\mp,E})(E) = 0$, that is, $\overline{\phi_\pm(E,x)}$ and $\phi_\pm(E,x)$ are linearly dependent, which is impossible at inner points of the set $\sigma_\pm^{(1)}$.

In summary, $\hat W(\lambda) \neq 0$ for $\lambda \in \mathrm{int}(\sigma) \setminus (\partial\sigma_-^{(1)} \cap \partial\sigma_+^{(1)})$ which finishes the first part and it remains to investigate the order of zeros.

Let $E \in \partial\sigma \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)})$ (these are the points of type E_1, E_3 and E_4 from our example). The function $\hat W(\lambda)$ is continuously differentiable with respect to the local parameter τ. Since at E we have $\frac{d}{d\tau}(\delta_+\delta_-)(E) = 0$, the function $W(\hat\phi_+,\hat\phi_-)$ has the same order of zero at E as $\hat W(\lambda)$. But if $\hat\delta(E) \neq 0$, then $\frac{d}{d\tau}\hat\delta_\pm(E) = 0$ and if $\hat\delta_-(E) = \hat\delta_+(E) = 0$, then $\frac{d}{d\tau}(\tau^{-2}\,\hat\delta_+\,\hat\delta_-)(E) = 0$. Therefore $\frac{d}{d\tau}\hat W(E) = 0$ if and only if $\frac{d}{d\tau}W(\hat\phi_{+,E},\hat\phi_{-,E}) = 0$.

To simplify notations, we will just write $\hat\phi_\pm := \hat\phi_{\pm,E}$ until the end of this proof. Again we have consider all possible cases for the mutual location of the Dirichlet eigenvalues.

First let $E \in \partial\sigma^{(2)} \cap \partial\sigma$ (a point of type E_2 in our example) and let $E \notin (\hat M_+ \cup \hat M_-)$. Then $\hat W(E) = 0$ if and only if (see (3.9)) $W(E) = W(\phi_+,\phi_-) = 0$, that is, $\phi_\pm(E,\cdot) = c_\pm\,\phi_\mp(E,\cdot)$, $c_-c_+ = 1$, $c_-, c_+ \in \mathbb{R}$. The derivative of the Jost solution with respect to τ is again a solution of equation (3.1). Therefore, by Lemma B.1,

$$\text{(B.6)}\qquad \begin{aligned}\dot W(E) &= W(\dot\phi_+,\phi_-) - W(\dot\phi_-,\phi_+) = c_-\,W(\dot\phi_+,\phi_+) - c_+\,W(\dot\phi_-,\phi_-)\\ &= c_-\,W(\dot\psi_+,\psi_+) - c_+\,W(\dot\psi_-,\psi_-) = -(c_+\,d_- + c_-\,d_+),\end{aligned}$$

where

$$d_\pm = \lim_{\lambda\to E} \frac{\mathrm{i}}{2g_\pm(\lambda)\sqrt{\lambda - E}}.$$

We see from (2.6) that $d_\pm \in \mathrm{i}\mathbb{R}_+ \setminus \{0\}$ if E is a left edge of σ and $d_\pm \in \mathbb{R}_+ \setminus \{0\}$ if E is a right edge of σ. Since $\operatorname{sign} c_- = \operatorname{sign} c_+$, this finishes the case $E \in \partial\sigma^{(2)} \cap \partial\sigma \setminus (\hat{M}_+ \cup \hat{M}_-)$.

The same arguments are valid for $E \in \partial\sigma^{(2)} \cap \partial\sigma \cap \hat{M}_+ \cap \hat{M}_-$ and $\hat{W}(E) = 0$. Then $\hat{\phi}_\pm(E, \cdot\,) = c_\pm\, \hat{\phi}_\mp(E, \cdot\,)$ and using Lemma B.3 we conclude that $c_-, c_+ \in \mathbb{R}$, $\operatorname{sign} c_- = \operatorname{sign} c_+$. By Lemma B.1

$$\frac{d}{d\tau} W(\hat{\phi}_+, \hat{\phi}_-)(E) = c_+ \lim_{\lambda\to E} \frac{\sqrt{\lambda - E}}{2g_-(\lambda)} + c_- \lim_{\lambda\to E} \frac{\sqrt{\lambda - E}}{2g_+(\lambda)} \neq 0.$$

The cases, when one of the Jost solutions is bounded in the edge of spectrum, and the another one is unbounded, are more subtle. For example, let $\hat{W}(E) = 0$ for $E \in (\hat{M}_+ \cap \partial\sigma^{(2)} \cap \partial\sigma) \setminus (\hat{M}_- \cap \hat{M}_+)$ and let E be a right band edge (like the point E_2 of or example when $\mu_+ = E_2$ and $\mu_- \neq E_2$). Then by Lemma B.3 $\hat{\phi}_+(x, E) \in \mathbb{R}$ and $\hat{\phi}_-(x, E) = \phi_-(x, E) \in \mathbb{R}$. Also $\frac{d}{d\tau}\hat{W}(E) \neq 0$ if and only if $\frac{d}{d\tau} W(\hat{\phi}_+,\, \phi_-)(E) \neq 0$. Therefore we have $\hat{\phi}_+ = c_+ \phi_-$, $\phi_- = c_- \hat{\phi}_+$ and $\operatorname{sign} c_+ = \operatorname{sign} c_-$. Thus, by Lemma B.1 $\frac{d}{d\tau} W(\hat{\phi}_+,\, \phi_-)(E) = c_- d_+ - c_+\, d_-$, where

$$d_+ = \lim_{\lambda\to E} \frac{\mathrm{i}\sqrt{\lambda - E}}{2\mathrm{i}g_+(\lambda)} \in \mathbb{R}_- \setminus \{0\}, \quad d_- = \lim_{\lambda\to E} \frac{-1}{\mathrm{i}\sqrt{\lambda - E}\, 2\mathrm{i}g_-(\lambda)} \in \mathbb{R}_+ \setminus \{0\}.$$

The case, when $E \in (\hat{M}_- \cap \partial\sigma^{(2)} \cap \partial\sigma) \setminus (\hat{M}_+ \cap \hat{M}_-)$ and E is a right band edge is analogous.

Now, let $E \in (\hat{M}_+ \cap \partial\sigma^{(2)} \cap \partial\sigma) \setminus \hat{M}_-$ and let E be a left band edge. Then by Lemma B.3 $\hat{\phi}_+ \in \mathrm{i}\mathbb{R}$ and $c_+, c_- \in \mathrm{i}\mathbb{R}$, $c_+\, c_- = 1$, that is, $\operatorname{sign}(\mathrm{i}c_+) = -\operatorname{sign}(\mathrm{i}c_-)$. Furthermore, $\operatorname{sign}\sqrt{\lambda - E} > 0$ since $\lambda > E$, and $\operatorname{sign} g_+(\lambda) = \operatorname{sign} g_-(\lambda)$. Therefore,

$$\frac{d}{d\tau} W(\hat{\phi}_+, \phi_-)(E) = \mathrm{i}c_- \lim_{\lambda\to E} \frac{\sqrt{\lambda - E}}{g_+(\lambda)} - \mathrm{i}\; c_+ \lim_{\lambda\to E} \frac{1}{\sqrt{\lambda - E}\, g_-(\lambda)} \neq 0.$$

Unlike the case of one and the same background $p_+(x) = p_-(x)$, where $\hat{W}(\lambda) \neq 0$ for $\lambda \in \operatorname{int}(\sigma)$, we could have $\hat{W}(\lambda) = 0$ for $\lambda \in \operatorname{int}(\sigma)$ in our steplike situation. The points under consideration are points of the set $\partial\sigma_-^{(1)} \cap \partial\sigma_+^{(1)}$. This case can be treated in the same way as the case $E \in \partial\sigma^{(2)} \cap \partial\sigma$. Namely, if $E \notin (\hat{M}_- \cup \hat{M}_+)$ then observe that one of the two summands in (B.6) is real and the other one is imaginary. The same is valid for the case $E \in (\hat{M}_- \cap \hat{M}_+)$. Now let $E \in \partial\sigma_-^{(1)} \cap \partial\sigma_+^{(1)} \cap \hat{M}_+ \setminus (\hat{M}_+ \cap \hat{M}_-)$ and let E be a right band edge of $\sigma_-^{(1)}$ (i.e. a left band edge of $\sigma_+^{(1)}$). Then by Lemma B.3 $\hat{\phi}_+(x, E) \in \mathrm{i}\mathbb{R}$ and $\hat{\phi}_-(x, E) = \phi_-(x, E) \in \mathbb{R}$. Moreover, $\hat{W} = 0$ if and only if $W(\hat{\phi}_+, \hat{\phi}_-) = 0$ with the same order of zero. Therefore, $c_+, c_- \in \mathrm{i}\mathbb{R}$ and by (B.2) and (2.14)

$$\begin{aligned} \frac{d}{d\tau} W(\hat{\phi}_+, \hat{\phi}_-) &= c_- W(\dot{\hat{\psi}}_+, \hat{\psi}_+) - c_+ W(\dot{\hat{\psi}}_-, \hat{\psi}_-) \\ &= \frac{1}{2\pi}\Big(\mathrm{i}c_- \lim_{\lambda\to E} \frac{\tau}{g_+} + \mathrm{i}c_+ \lim_{\lambda\to E} \frac{1}{\tau g_-}\Big) \neq 0 \end{aligned}$$

because the first summand is imaginary and the second one is real. All other combinations can be treated similarly.

Finally, consider the case $E \in \partial\sigma_\pm^{(1)} \cap \partial\sigma$. Let, for example, $E \in \partial\sigma_+^{(1)} \cap \partial\sigma$ (the point E_4 in our example). Since $E \in \mathbb{R} \setminus \sigma_-$ in this case, we have $\hat{\delta}_-(E) \neq 0$

and one has to study zero of the Wronskian $W(\hat{\phi}_+, \tilde{\phi}_-)$. But by (2.8) and (3.2) $W\big(\tilde{\phi}_-, \frac{d}{d\tau}\tilde{\phi}_-\big) = W\big(\tilde{\psi}_-, \frac{d}{d\tau}\tilde{\psi}_-\big) = 0$. Therefore,

$$\frac{d}{d\tau}W(\hat{\phi}_+, \tilde{\phi}_-) = c_- W\Big(\frac{d}{d\tau}\hat{\phi}_+, \hat{\phi}_+\Big) \neq 0$$

by Lemma B.1. □

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Institut de Mathématiques de Jussieu, case 7012, Université Paris 7, 2 place Jussieu, 75251 Paris, France

E-mail address: `aboutet@math.jussieu.fr`

B.Verkin Institute for Low Temperature Physics, 47 Lenin Avenue, 61103 Kharkiv, Ukraine

E-mail address: `egorova@ilt.kharkov.ua`

Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria

E-mail address: `Gerald.Teschl@univie.ac.at`

URL: `http://www.mat.univie.ac.at/~gerald/`